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Algebraic treatment of some systems with spin-like interactions with applications in quantum optics and vibronic spectroscopy

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Abstract

A unified scheme based on algebraic techniques is proposed to solve the eigenvalue equation for a class of systems involving spin-like interactions. From general assumptions Hamiltonian models are built from selected elements in the enveloping algebra of the harmonic oscillator, $su(2)$ and $su(1,1)$ algebras. Our method is next illustrated through examples taken in the areas of quantum optics and dynamical Jahn–Teller systems in orbital doublets.

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1. Introduction

The determination of exactly solvable models and their physical realizations has been a subject of constant interest since the very beginning of quantum mechanics. Quite often they can be used as zeroth-order approximations of more complicated systems and furnish the necessary ingredients for a perturbative treatment of higher-order interactions. They usually reveal what has been called hidden symmetries which, besides their own mathematical interest, provide tools for practical computations.

Among the various methods which have been used to build and treat exactly solvable models the spectrum generating algebra (SGA) formalism [1–3] and the closely related concept of non-invariance dynamical group [4–7] appear as the most fruitful, especially in situations where several degrees of freedom are involved. Various definitions have been given for a SGA [1, 2, 4, 8] and extensions have been proposed, for instance, with the concept of generalized Lie algebra [9, 10] in order to generate spectra of nonlinear Hamiltonians. These extensions also allow one to treat in a unified way several problems which otherwise would not enter the standard scheme.

In the spirit of these algebraic methods we consider here a class of systems for which we assume a 'dominant' spin-like (or pseudo-spin) interaction written in the form

$$\mathcal{O} = S_+ A + S_- A^\dagger, \tag{1}$$

where $S_\pm = S_x \pm \mathrm{i}S_y$, S_z are generators for a spin 1/2 algebra. The A operators will be chosen in several manners, the main constraint being that from the set S_+A, $S_-A^\dagger$, $[S_+A, S_-A^\dagger]$ we should be able to build a spin 1/2 algebra denoted $su(2)(P)$ in the following. The form of the A operator determines also some conserved quantities and we can build Hamiltonian models with the requirement that they can be brought to diagonal form through a unitary transformation (or rotation) which is an element of the $SU(2)(P)$ group.

In section 2 the basis of our method is presented. The A operators are taken as particular elements in the enveloping algebras of the harmonic oscillator algebra h_4, $su(2)$ and $su(1, 1)$. The $su(2)(P)$ algebras obtained appear, then, as generalizations of the $su(2)$ algebra introduced [11, 12] for one-mode Jaynes–Cummings models (JCMs). In each case we determine general expressions of solvable Hamiltonian models. Their eigenvalues are calculated and their eigenvectors completely determined through a unitary transformation of $SU(2)(P)$.

In our applications (section 3) we first indicate briefly how known results for time-independent one-mode JCMs [13–15] can be recovered. Other examples concerning two-level systems, which are all treated in a unified way, are more detailed when our method offers a new solution or extend previous results. This is the case, for instance, for the modified two-mode JCM [16–18] which is treated in a new way and generalized to a p-mode case. Finally, exact solutions are obtained for several Jahn–Teller Hamiltonians in orbital doublets.

In the last section and through examples we show that more general A operators may be introduced which shows the range of potential applications of our method.

2. Construction of $su(2)(P)$ algebras and Hamiltonian models

In the hermitian operator $\mathcal{O}$ (1) we first assume that A is a power in one of the ladder operators for one of the Lie algebras $\mathcal{A} = h_4$, $su(2)$ or $su(1, 1)$. The square of $\mathcal{O}$ writes

$$\begin{aligned}\mathcal{O}^2 &= \left\{\left(\tfrac{1}{2} + S_z\right)AA^\dagger + \left(\tfrac{1}{2} - S_z\right)A^\dagger A\right\} \\ &= \left\{\tfrac{1}{2}(A^\dagger A + AA^\dagger) + S_z[A, A^\dagger]\right\} = \mathcal{F},\end{aligned} \tag{2}$$

and is a positive operator, function of the invariant $\mathcal{I}$ and of the weight generator A_z of $\mathcal{A}$ and of the pseudo-spin component S_z. A basis for the space of states $\mathcal{H}$ is

$$|\Lambda\lambda\rangle\left|\tfrac{1}{2} \pm \tfrac{1}{2}\right\rangle \equiv |\Lambda\lambda\rangle|\pm\rangle, \tag{3}$$

where $|\Lambda\lambda\rangle$ is a basis for an irreducible representation (IR) of $\mathcal{A}$ and $|\pm\rangle$ eigenstates of S_z and

$$\mathcal{F}|\Lambda\lambda\rangle|\pm\rangle = f_\pm(\Lambda, \lambda)|\Lambda\lambda\rangle|\pm\rangle. \tag{4}$$

We have $\mathcal{H} = \mathcal{H}_n \oplus \mathcal{H}_F$, where $\mathcal{H}_n$ is the null-space of $\mathcal{F}$ and also that of $\mathcal{O}$. With (2) and $S_z^2 = 1/4$ one obtains the following properties:

$$[\mathcal{F}, S_+A] = [\mathcal{F}, S_-A^\dagger] = 0, \qquad [S_+A, S_-A^\dagger] = 2S_z\mathcal{F}. \tag{5}$$

We may then build the operator algebra acting in $\mathcal{H}_F$,

$$P_+ = \frac{1}{\sqrt{\mathcal{F}}}S_+A, \qquad P_- = \frac{1}{\sqrt{\mathcal{F}}}S_-A^\dagger, \qquad P_z = S_z, \tag{6}$$

which is easily shown to be isomorphic to a $su(2)$ spin 1/2 algebra:

$$[P_z, P_\pm] = \pm P_\pm, \qquad [P_+, P_-] = 2P_z, \qquad \tfrac{1}{2}(P_+P_- + P_-P_+) + P_z^2 = 3/4. \tag{7}$$

Table 1. Generators for some spin 1/2 algebras (equation (11)).

Algebra	$\mathcal{F}$	$su(2)(M)$	$\overline{\mathcal{F}}$	$su(2)(N)$
h_4	$a^+a + \frac{1}{2} + S_z$	$M_+ = (\mathcal{F})^{-1/2} S_+ a$	$a^+a + \frac{1}{2} - S_z$	$N_+ = (\overline{\mathcal{F}})^{-1/2} S_+ a^+$
		$M_- = (\mathcal{F})^{-1/2} S_- a^+$		$N_- = (\overline{\mathcal{F}})^{-1/2} S_- a$
$su(2)$	$J^2 - J_z^2$	$M_+ = (\mathcal{F})^{-1/2} S_+ J_-$	$J^2 - J_z^2$	$N_+ = (\overline{\mathcal{F}})^{-1/2} S_+ J_+$
	$-2S_z J_z$	$M_- = (\mathcal{F})^{-1/2} S_- J_+$	$+2S_z J_z$	$N_- = (\overline{\mathcal{F}})^{-1/2} S_- J_-$
$su(1,1)$	$-K^2 + K_z^2$	$M_+ = (\mathcal{F})^{-1/2} S_+ K_-$	$-K^2 + K_z^2$	$N_+ = (\overline{\mathcal{F}})^{-1/2} S_+ K_+$
	$+2S_z K_z$	$M_- = (\mathcal{F})^{-1/2} S_- K_+$	$-2S_z K_z$	$N_- = (\overline{\mathcal{F}})^{-1/2} S_- K_-$

To simplify and since there will be no ambiguities we use the same notation for the $\mathcal{F}$ operator acting in $\mathcal{H}$ and its restriction onto $\mathcal{H}_F$. Each subspace $\mathcal{H}_{\Lambda\lambda} \times \mathcal{H}_{1/2}$ of $\mathcal{H}_F$ is thus split into two subspaces $\mathcal{H}_+$ ($\mathcal{H}_-$) with bases $|\Lambda\lambda\rangle|+\rangle$ ($|\Lambda\lambda\rangle|-\rangle$) with the properties

$$\begin{aligned} &\mathcal{F}|\Lambda\lambda\rangle|\pm\rangle = f_\pm(\Lambda,\lambda)|\Lambda\lambda\rangle|\pm\rangle, \qquad f_\pm(\Lambda,\lambda) > 0 \qquad P_\pm|\Lambda\lambda\rangle|\pm\rangle = 0, \\ &P_\mp|\Lambda\lambda\rangle|\pm\rangle = |\Lambda\lambda'\rangle|\mp\rangle, \qquad f_\pm(\Lambda,\lambda) = f_\mp(\Lambda,\lambda'), \end{aligned} \tag{8}$$

and each nonzero eigenvalue of $\mathcal{F}$ is at least doubly degenerate. In the simplest case the A operator is the ladder operator itself and we have two possibilities:

$$\mathcal{O} = S_+X_- + S_-X_+, \qquad \overline{\mathcal{O}} = S_+X_+ + S_-X_-, \tag{9}$$

where X_- (resp. X_+) is the lowering generator (resp. raising generator) of $\mathcal{A}$. For these cases, we have

$$\begin{aligned} \mathcal{F} &= \left\{\left(\tfrac{1}{2} + S_z\right)X_-X_+ + \left(\tfrac{1}{2} - S_z\right)X_+X_-\right\} = \left\{\tfrac{1}{2}(X_+X_- + X_-X_+) + S_z[X_-, X_+]\right\}, \\ \overline{\mathcal{F}} &= \left\{\left(\tfrac{1}{2} + S_z\right)X_+X_- + \left(\tfrac{1}{2} - S_z\right)X_-X_+\right\} = \left\{\tfrac{1}{2}(X_+X_- + X_-X_+) - S_z[X_-, X_+]\right\}, \end{aligned} \tag{10}$$

and we obtain two spin 1/2 algebras

$$\begin{aligned} M_+ &= \frac{1}{\sqrt{\mathcal{F}}}S_+X_-, \qquad M_- = \frac{1}{\sqrt{\mathcal{F}}}S_-X_+, \qquad M_z = S_z, \\ N_+ &= \frac{1}{\sqrt{\overline{\mathcal{F}}}}S_+X_+, \qquad N_- = \frac{1}{\sqrt{\overline{\mathcal{F}}}}S_-X_-, \qquad N_z = S_z. \end{aligned} \tag{11}$$

The operators $M_\pm, N_\pm, \mathcal{F}, \overline{\mathcal{F}}$ are given in table 1 for h_4, $su(2)$ and $su(1,1)$.

The interaction operator $\mathcal{O}$ (1) can thus be expressed in terms of a 'constant of motion' $\mathcal{F}$ and of the $su(2)(P)$ generators. These results allow first to build a generic hermitian Hamiltonian model

$$H = H_0 + \delta S_+A + \delta^* S_-A^\dagger + \delta_z S_z = H_0 + \sqrt{\mathcal{F}}(\delta P_+ + \delta^* P_-) + \delta_z P_z, \tag{12}$$

where by H_0 we denote an operator which commutes with S_+A, $S_-A^\dagger$ and S_z. This model can be diagonalized through a standard unitary transformation of $SU(2)(P)$ [19, 20]. Setting $\delta = |\delta| \exp(\mathrm{i}\varphi)$, we obtain

$$U = \exp(\xi P_+ - \xi^\dagger P_-), \tag{13}$$

with

$$\xi = |\xi| \exp(\mathrm{i}\varphi), \qquad \tan|\xi| = \left(\frac{\Omega(\mathcal{F}) - \delta_z/2}{\Omega(\mathcal{F}) + \delta_z/2}\right)^{1/2}, \qquad \Omega(\mathcal{F}) = \left[\mathcal{F}|\delta|^2 + \frac{\delta_z^2}{4}\right]^{1/2}, \tag{14}$$

and

$$U H U^{-1} = \widetilde{H} = H_0 + \epsilon_z S_z = H_0 + 2\Omega(\mathcal{F}) S_z. \tag{15}$$

Denoting $|\Psi_\pm\rangle$ the degenerate eigenstates of H_0 associated with the eigenvalue f of $\mathcal{F}$, we have

$$\widetilde{H}|\Psi_\pm\rangle = [E_0 \pm \Omega(f)]|\Psi_\pm\rangle, \qquad \Omega(f) = \left[f|\delta|^2 + \frac{\delta_z^2}{4} \right]^{1/2}, \tag{16}$$

and the eigenstates of H (12) are $|\widetilde{\Psi}_\pm\rangle = U^{-1}|\Psi_\pm\rangle$. With the $SU(2)$ disentangling formula [19] and the properties of spin 1/2 operators we obtain

$$\begin{aligned} |\widetilde{\Psi}_\pm\rangle &= U^{-1}|\Psi_\pm\rangle = \exp[-(\xi P_+ - \xi^\dagger P_-)]|\Psi_\pm\rangle \\ &= \exp(k_-^\dagger P_+) \exp(k_z P_z) \exp(-k_- P_-)|\Psi_\pm\rangle \\ &= (I + k_-^\dagger P_+) \exp(k_z P_z)(I - k_- P_-)|\Psi_\pm\rangle \\ &= (I - k_- P_-) \exp(-k_z P_z)(I + k_-^\dagger P_+)|\Psi_\pm\rangle, \end{aligned} \tag{17}$$

where the last two forms are useful to obtain explicitly the states $|\widetilde{\Psi}_\pm\rangle$ and with

$$k_z = \ln\left[\frac{2\Omega(\mathcal{F})}{\Omega(\mathcal{F}) + \delta_z/2}\right], \qquad k_- = -\frac{(\Omega(\mathcal{F}) - \delta_z/2)}{\delta\sqrt{\mathcal{F}}} = -\exp(-\mathrm{i}\varphi)\left(\frac{\Omega(\mathcal{F}) - \delta_z/2}{\Omega(\mathcal{F}) + \delta_z/2}\right)^{1/2}. \tag{18}$$

The states (17) can also be written as

$$\begin{aligned} |\widetilde{\Psi}_+\rangle &= \cos[\theta(\mathcal{F})]|\Psi_+\rangle + \exp(-\mathrm{i}\varphi)\sin[\theta(\mathcal{F})]|\Psi_-\rangle, \\ |\widetilde{\Psi}_-\rangle &= \cos[\theta(\mathcal{F})]|\Psi_-\rangle - \exp(\mathrm{i}\varphi)\sin[\theta(\mathcal{F})]|\Psi_+\rangle, \end{aligned} \tag{19}$$

with

$$\cos[\theta(\mathcal{F})] = \left[\frac{\Omega(\mathcal{F}) + \delta_z/2}{2\Omega(\mathcal{F})}\right]^{1/2}, \qquad \sin[\theta(\mathcal{F})] = \left[\frac{\Omega(\mathcal{F}) - \delta_z/2}{2\Omega(\mathcal{F})}\right]^{1/2}. \tag{20}$$

We note that, in the special case $\delta_z = 0$, the preceding relations are valid and U^{-1} takes the simple form

$$U^{-1} = \exp\left[-\frac{\pi}{4}(\mathrm{e}^{\mathrm{i}\varphi} P_+ - \mathrm{e}^{-\mathrm{i}\varphi} P_-)\right] = \exp(-\mathrm{e}^{\mathrm{i}\varphi} P_+) \exp[\ln(2) P_z] \exp(\mathrm{e}^{-\mathrm{i}\varphi} P_-), \tag{21}$$

with $\Omega(\mathcal{F}) = \sqrt{\mathcal{F}}|\delta|$ and $\cos[\theta(\mathcal{F})] = \sin[\theta(\mathcal{F})] = 1/\sqrt{2}$.

More elaborate models are obtained specifying A operators and considering various possible forms for H_0. Basically we have two main cases according as A in (1) is a power in the lowering or raising operator of $\mathcal{A}$ ($p \in \mathbb{N}$, $p > 0$):

$$\begin{array}{lll} \text{Case (i)} & \text{Case (ii)} & \\ A = \rho(N) a^p & A = a^{+p} \rho(N) & h_4 \\ A = \rho(J_z) J_-^p & A = J_+^p \rho(J_z) & su(2) \\ A = \rho(K_z) K_-^p & A = K_+^p \rho(K_z) & su(1,1) \end{array} \tag{22}$$

where $\rho(x)$ is an entire function of x which, for simplicity, we take as real. Here and in the following, additional dependences of operators upon the invariant $\mathcal{I}$ of $\mathcal{A}$ are implied.

We note that, in the h_4 case, our choice for the A operators contains the generalized Bose operators defined by Brandt *et al* [21] and further extended in [22] (for a review, see [23] and references therein); however, we do not impose the constraint $[A, A^\dagger] = I$. Also several deformation schemes of these classical algebras discussed in the literature [24–26] may be included.

For case (i), we have, with equation (6),

$$[A_z + pS_z, P_\pm] = 0, \qquad [A_z + pS_z, P_z] = 0, \tag{23}$$

and of course $[A_z, \mathcal{F}] = 0$ with $\mathcal{F}$ given by (2). Setting $\Delta = A_z + pS_z$ (or $\Delta = A_z \pm p/2 + pS_z$) we can thus consider Hamiltonian expansions of the form

$$\begin{aligned} H &= H_0 + H_0'(\Delta) + H_0''(A_z) + \delta S_+ A + \delta^* S_- A^\dagger + \gamma_z(A_z) S_z \\ &= H_0 + H_0'(\Delta) + H_0''(A_z) + \sqrt{\mathcal{F}}(\delta P_+ + \delta^* P_-) + \gamma_z(A_z) P_z. \end{aligned} \tag{24}$$

In fact, in all three cases, it may be shown that for any expandable function Φ and with $\Delta = A_z + p\left(\frac{1}{2} + S_z\right)$ we have

$$\begin{aligned} \Phi(\Delta) &= \left(\tfrac{1}{2} - S_z\right)\Phi(A_z) + \left(\tfrac{1}{2} + S_z\right)\Phi(A_z + p), \\ \Phi(A_z) &= \left(\tfrac{1}{2} - S_z\right)\Phi(\Delta) + \left(\tfrac{1}{2} + S_z\right)\Phi(\Delta - p), \end{aligned} \tag{25}$$

with similar expressions for other choices of Δ. This allows us to rewrite (24) more simply in terms of the conserved quantities Δ and $\mathcal{F}$:

$$H = H_0 + H'(\Delta) + \sqrt{\mathcal{F}}(\delta P_+ + \delta^* P_-) + \delta_z(\Delta) P_z. \tag{26}$$

From equations (2), (22) the $\mathcal{F}$ operators are given by:

- For the oscillator algebra

$$\begin{aligned} \mathcal{F} &= \left(\tfrac{1}{2} + S_z\right)\rho(N) a^p a^{+p} \rho(N) + \left(\tfrac{1}{2} - S_z\right) a^{+p} \rho^2(N) a^p \\ &= \left(\tfrac{1}{2} + S_z\right)\rho^2(N)(N+p)^{\underline{p}} + \left(\tfrac{1}{2} - S_z\right)\rho^2(N-p) N^{\underline{p}}, \end{aligned} \tag{27}$$

with $N^{\underline{p}} = N(N-1)\cdots(N-p+1)$, $N^{\underline{0}} = 1$. With $|n\rangle$ denoting the usual Fock states the degenerate eigenstates (8) are $|n\rangle|+\rangle$, $|n+p\rangle|-\rangle$ associated with the eigenvalues

$$f(n) = \rho^2(n)(n+p)^{\underline{p}} = \rho^2(n)\frac{(n+p)!}{n!}, \qquad \rho(n) \neq 0, \tag{28}$$

of $\mathcal{F}$ and $\kappa = n + p$ of $\Delta = a^+ a + p(1/2 + S_z)$.

- For $su(2)$

$$\begin{aligned} \mathcal{F} &= \left(\frac{1}{2} + S_z\right)\rho(J_z) J_-^p J_+^p \rho(J_z) + \left(\frac{1}{2} - S_z\right) J_+^p \rho^2(J_z) J_-^p \\ &= \left(\frac{1}{2} + S_z\right)\rho^2(J_z)\prod_{u=0}^{p-1}\{J^2 - (J_z + u + 1)(J_z + u)\} \\ &\quad + \left(\frac{1}{2} - S_z\right)\rho^2(J_z - p)\prod_{u=0}^{p-1}\{J^2 - (J_z - u)(J_z - u - 1)\}. \end{aligned} \tag{29}$$

A basis for the unitary IRs of $su(2)$ is given by the standard $su(2) \supset so(2)$ basis $\{|jm\rangle\}$ corresponding to the eigenvalues $j(j+1)$ of the Casimir operator $J^2 = (J_+J_- + J_-J_+)/2 + J_z^2$ and m of J_z. With $p \leqslant 2j$, the degenerate eigenstates $|jm\rangle|+\rangle$, $|jm+p\rangle|-\rangle$ $(j - m \geqslant p)$ are associated with the eigenvalues $\kappa = m + p$ of $\Delta = J_z + p(1/2 + S_z)$ and $f(j, m)$ of $\mathcal{F}$:

$$f(j, m) = \rho^2(m)\frac{(j-m)!(j+m+p)!}{(j+m)!(j-m-p)!}, \qquad \rho(m) \neq 0. \tag{30}$$

- For $su(1,1)$

$$\mathcal{F} = \left(\frac{1}{2}+S_z\right)\rho(K_z)K_-^pK_+^p\rho(K_z) + \left(\frac{1}{2}-S_z\right)K_+^p\rho^2(K_z)K_-^p$$
$$= \left(\frac{1}{2}+S_z\right)\rho^2(K_z)\prod_{u=0}^{p-1}\{(K_z+u)(K_z+u+1)-K^2\}$$
$$+\left(\frac{1}{2}-S_z\right)\rho^2(K_z-p)\prod_{u=0}^{p-1}\{(K_z-u)(K_z-u-1)-K^2\}. \qquad (31)$$

For this algebra, we consider only the positive discrete series IRs $\mathcal{D}^+(k)$ [1] for which the Casimir operator $K^2 = K_z^2 - (K_+K_- + K_-K_+)/2$ together with the compact generator K_z are diagonal:

$$K^2|km\rangle = k(k-1)|km\rangle \quad k>0, \qquad K_z|km\rangle = (k+m)|km\rangle \quad m = 0,1,\ldots,\infty.$$

The degenerate eigenstates (8) are $|km\rangle|+\rangle$, $|km+p\rangle|-\rangle$ associated with the eigenvalues $\kappa = m+k+p$ of $\Delta = K_z + p(1/2+S_z)$ and $f(k,m)$ of $\mathcal{F}$:

$$f(k,m) = \rho^2(m+k)\frac{(m+p)!\Gamma(m+2k+p)}{(m)!\Gamma(m+2k)}, \qquad \rho(m+k)\neq 0. \qquad (32)$$

In each case the subspace $\mathcal{H}_n$ is determined by the zero eigenvalues of $\mathcal{F}$ and is also associated with uncoupled states by the interaction term $\delta S_+A + \delta^*S_-A^\dagger$ in the Hamiltonian (24). We thus have the states $|\Lambda\lambda\rangle|\pm\rangle$ for which

$$\rho(A_z)|\Lambda\lambda\rangle|\pm\rangle = 0,$$

and for h_4 the states $|n\rangle|-\rangle$ with $n : 0,1,\ldots,p-1$. For $su(2)$, besides those states for which $p > 2j$ we find $|jm\rangle|+\rangle$ with $m = j-p+1, j-p+2,\ldots,j$ and $|jm\rangle|-\rangle$ with $m = -j, -j+1,\ldots,-j+p-1$. For $su(1,1)$, as for h_4, there is no upper bound on the m value so the uncoupled states are $|km\rangle|-\rangle$ with $m = 0,1,\ldots,p-1$.

For case (ii), we have, with equations (6), (22),

$$[A_z - pS_z, P_\pm] = 0, \qquad [A_z - pS_z, P_z] = 0, \qquad (33)$$

and still $[A_z, \overline{\mathcal{F}}] = 0$ with $\overline{\mathcal{F}}$ given by (2). Setting $\overline{\Delta} = A_z - pS_z$ (or $\overline{\Delta} = A_z \pm p/2 - pS_z$), we can consider Hamiltonian expansions of the form

$$H = H_0 + H_0'(\overline{\Delta}) + H_0''(A_z) + \delta S_+A + \delta^*S_-A^\dagger + \gamma_z(A_z)S_z$$
$$= H_0 + +H_0'(\overline{\Delta}) + H_0''(A_z) + \sqrt{\overline{\mathcal{F}}}(\delta P_+ + \delta^*P_-) + \gamma_z(A_z)P_z. \qquad (34)$$

Choosing $\overline{\Delta} = A_z + p\left(\frac{1}{2} - S_z\right)$ equation (25) keeps the same form with the substitutions $\Delta \to \overline{\Delta}$ and the interchange $1/2 - S_z \leftrightarrow 1/2 + S_z$. Then we also have for (34)

$$H = H_0 + H'(\overline{\Delta}) + \sqrt{\overline{\mathcal{F}}}(\delta P_+ + \delta^*P_-) + \delta_z(\overline{\Delta})P_z. \qquad (35)$$

The operators $\overline{\mathcal{F}}$ are determined from equations (2), (22) or directly from those of case (i) with the interchange $1/2 - S_z \leftrightarrow 1/2 + S_z$ in equations (27), (29), (31). The degenerate eigenstates are obtained from those of case (i) with the interchange $|+\rangle \leftrightarrow |-\rangle$, the eigenvalues $\bar{\kappa}$ of $\overline{\Delta}$ and $\bar{f}$ of $\overline{\mathcal{F}}$ being identical to those κ of Δ and f of $\mathcal{F}$ (equations (28), (30), (32)). The same rule holds for the uncoupled states discussed before.

The form of the Hamiltonians (26), (35) allows us to obtain their eigenspectrum with a straightforward adaptation of the results in equations (12)–(20):

$$UHU^{-1} = \widetilde{H} = H_0 + H'(\Delta) + \epsilon_z S_z = H_0 + H'(\Delta) + 2\Omega(\mathcal{F},\Delta)S_z, \qquad (36)$$

with

$$\Omega(\mathcal{F}, \Delta) = \left[\mathcal{F}|\delta|^2 + \frac{\delta_z^2(\Delta)}{4}\right]^{1/2}, \tag{37}$$

and eigenvalues determined by

$$\widetilde{H}|\Psi_\pm\rangle = [E_0 + E'(\kappa) \pm \Omega(f, \kappa)]|\Psi_\pm\rangle. \tag{38}$$

Equations (13),(14) and (17)–(20) keep the same form with the substitutions $\Omega(\mathcal{F}) \to \Omega(\mathcal{F}, \Delta)$, $\delta_z \to \delta_z(\Delta)$ and $\theta(\mathcal{F}) \to \theta(\mathcal{F}, \Delta)$. Alternatively we can take $\theta(\mathcal{F}, \Delta) \equiv \theta(f, \kappa)$ in equation (19). We note that, taking into account the relation

$$2S_z = \left(\tfrac{1}{2} + S_z\right) - \left(\tfrac{1}{2} - S_z\right) = S_+S_- - S_-S_+,$$

and the expressions (2) for $\mathcal{F}$ and (25) for $\delta_z(\Delta)$, the last term in (36) may be written as

$$\epsilon_z S_z = \Omega_+ S_+ S_- - \Omega_- S_- S_+, \tag{39}$$

where $\Omega_\pm$ may be seen as 'generalized flipping operators' [15, 27],

$$\Omega_\pm = \left[\mathcal{F}_\pm|\delta|^2 + \frac{\delta_{z\pm}^2(A_z)}{4}\right]^{1/2}, \tag{40}$$

with

$$\mathcal{F}_+ = AA^\dagger, \qquad \mathcal{F}_- = A^\dagger A, \qquad \delta_{z+}(A_z) = \delta_z(A_z + p), \qquad \delta_{z-}(A_z) = \delta_z(A_z). \tag{41}$$

Relations (36)–(41) have been detailed for case (i); the corresponding equations for case (ii) are obtained with the substitutions $\mathcal{F} \to \overline{\mathcal{F}}$, $\Delta \to \overline{\Delta}$ and equation (41) replaced by

$$\overline{\mathcal{F}_+} = AA^\dagger, \qquad \overline{\mathcal{F}_-} = A^\dagger A, \qquad \bar{\delta}_{z+}(A_z) = \delta_z(A_z), \qquad \bar{\delta}_{z-}(A_z) = \delta_z(A_z + p). \tag{42}$$

We show in the following section that our approach allows us to gather in a unified formalism various problems in quantum optics and molecular spectroscopy.

3. Applications

3.1. Single-mode JCMs

In the area of quantum optics the interaction between a two-level atom and a quantized single-mode electromagnetic field is described by the JCM [13–15]. Since its original formulation various generalizations and extensions have been proposed leading, for instance, to what is referred to as the nonlinear time-independent two-level multiphoton JCM including intensity-dependent coupling [28]. These have been studied by numerous authors from many different points of view. The relevance of the $su(2)$ algebra for these two-level models has already been recognized [11, 12, 27] and extended to $su(N)$ in some N-level cases [29]. Also using deformations of the oscillator algebra a unified description of one-mode JC Hamiltonians has been given [10, 30, 31]. Supersymmetric extensions and there deformed versions have also been explored [30, 32].

We just briefly sketch below how our Hamiltonian model (26) can be specialized to include all standard one-mode JCM and how our method gives a simple operator form for the unitary transformation to the eigenbasis. The standard one-mode multiphoton JCM may be written

$$H_{\mathrm{JC}}/\hbar = \omega a^+ a + \omega_0 S_z + \rho_0(a^+ a) + H_{\mathrm{int}}, \tag{43}$$

where the bosonic operators a, a^+ are associated with the radiation field with frequency ω; the pseudo-spin operators $\sigma_\pm = 2S_\pm$, $\sigma_z = 2S_z$ represent the two atomic levels separated by an energy gap of $\hbar\omega_0$. $\rho_0(a^+a)$ may be associated with the nonlinear effects of a Kerr-like medium [33, 34] usually written in the form $\chi a^{+2}a^2$. The atom–field interaction term $H_{\rm int}$ in the multiphoton case and within the rotating wave approximation may be taken as

$$H_{\rm int}/\hbar = g\rho(N)a^p S_+ + g^* a^{+p}\rho(N)S_-, \tag{44}$$

where $\rho(N)$ is a real analytic function of the photon number operator $N = a^+a$.

The correlation with the results established in section 2 is straightforward with $\mathcal{F}$ as given by (27), $\Delta = a^+a + p(1/2 + S_z)$ and (equation (26)):

$$H_0 = 0, \qquad H'(\Delta) = \omega\left(\Delta - \frac{p}{2}\right) + [\rho_0(\Delta) + \rho_0(\Delta - p)]/2,$$
$$\delta_z(\Delta) = [\omega_0 - p\omega + \rho_0(\Delta - p) - \rho_0(\Delta)], \qquad \delta = g.$$

This allows equations (36)–(38) to be used directly which gives, in particular, the energies

$$\begin{aligned} E_{\rm JC\pm} &= \hbar\left\{\omega\left(\kappa - \frac{p}{2}\right) + \frac{1}{2}[\rho_0(\kappa) + \rho_0(\kappa - p)] \pm \Omega(f,\kappa)\right\}, \\ \Omega(f,\kappa) &= \left[f|g|^2 + \frac{\delta_z^2(\kappa)}{4}\right]^{1/2}, \end{aligned} \tag{45}$$

with $\kappa = n + p$ and f as given by (28). Equations (17), (18) with $\Omega(\mathcal{F}) \to \Omega(\mathcal{F},\Delta)$ and $\delta_z \to \delta_z(\Delta)$ determine the unitary dressing operator U^{-1} from which the dressed states of $H_{\rm JC}$ may also be written in the form (equations (19), (20))

$$\begin{aligned} |\widetilde{\Psi}_+\rangle &= \left[\frac{\Omega(f) + \delta_z(\kappa)/2}{2\Omega(f)}\right]^{1/2}\left\{|n\rangle|+\rangle + \left[\frac{\Omega(f) - \delta_z(\kappa)/2}{g\sqrt{f}}\right]|n+p\rangle|-\rangle\right\} \\ &= \cos[\theta(f,\kappa)]|n\rangle|+\rangle + {\rm e}^{-{\rm i}\varphi}\sin[\theta(f,\kappa)]|n+p\rangle|-\rangle, \\ |\widetilde{\Psi}_-\rangle &= \left[\frac{\Omega(f) + \delta_z(\kappa)/2}{2\Omega(f)}\right]^{1/2}\left\{|n+p\rangle|-\rangle - \left[\frac{\Omega(f) - \delta_z(\kappa)/2}{g^*\sqrt{f}}\right]|n\rangle|+\rangle\right\} \\ &= \cos[\theta(f,\kappa)]|n+p\rangle|-\rangle - {\rm e}^{{\rm i}\varphi}\sin[\theta(f,\kappa)]|n\rangle|+\rangle, \end{aligned} \tag{46}$$

with $g = |g|\,{\rm e}^{{\rm i}\varphi}$. The null dressed states are $|n\rangle|-\rangle, n = 0, 1, \ldots, p-1$, with energies $E_{\rm JC0-} = \hbar(\omega n - \omega_0/2 + \rho_0(n))$. We note that, within the rotating wave approximation, only case (i) is relevant; more precisely, (ii) would correspond to a diagonalization of the counter-rotating terms. Several special cases can be underlined.

- For $p = 1$, $\rho_0(N) = 0$ and $\rho(N) = I$ then $\mathcal{F} = \Delta$ and we recover the original JCM [13].
- $p = 1$ and $\rho(N) = (2s + N)^{1/2}$ in (44) give the intensity-dependent coupling model [35, 36] associated with the usual Holstein–Primakoff realization of $su(1,1)$ [37, 38] and we can use equations (31), (32) with $\rho(K_z) = I$ and the correspondence between states:

  $$|n\rangle|+\rangle \to |k = s\, m = n\rangle|+\rangle, \qquad |n+1\rangle|-\rangle \to |k = s\, m+1 = n+1\rangle|-\rangle.$$
- The $p = 2$ case [39–41] can be equivalently treated with the one-mode realization of $su(1,1)$

  $$K_+ = a^{+2}/2, \qquad K_- = a^2/2, \qquad K_z = (a^+a + 1/2)/2,$$

  with $p = 1$ in equations (31), (32) and the correspondence between basis states

  $$|n\rangle|+\rangle \to |km\rangle|+\rangle, \qquad |n+2\rangle|-\rangle \to |km+1\rangle|-\rangle,$$

  with Bargmann index $k = \frac{1}{4}$ for $n = 2m$ and $k = \frac{3}{4}$ for $n = 2m+1$.
- As noted before, several multiphoton models of the one-mode JC type could be built starting from the generalized p-photon operators [21, 22], their deformed versions [24, 25] and the generalized Holstein–Primakoff realizations of $su(2)$ and $su(1,1)$ [42].

3.2. Multimode nonlinear JCMs

We consider below some problems in cavity quantum electrodynamics (QED) [43]. These are multiphoton two-mode models, of which only special cases have been treated in the literature. Next the three-mode nonlinear Raman coupled model and the modified two modes JCM are given a complete and simple solution, the latter being extended to a p-mode case. Finally, we show that several dynamical Jahn–Teller systems may be solved within the same formalism. We give in each case the effective Hamiltonian in a form which allows results in section 2 to be used and solve the eigenvalue equation.

3.2.1. Multiphoton two-mode models. Within our approach we can consider the following effective Hamiltonians for two-level systems involving multiphoton processes:

$$H/\hbar = \omega_0 S_z + \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \beta_1 S_- S_+ \rho_1\left(a_1^+ a_1\right) + \beta_2 S_+ S_- \rho_2(a_2^+ a_2) + H_{\text{int}}/\hbar,$$
$$H_{\text{int}}/\hbar = \begin{cases} \left[g\rho\left(a_1^+ a_1, a_2^+ a_2\right) a_2^{+p} a_1^p S_+ + g^* a_1^{+p} a_2^p \rho\left(a_1^+ a_1, a_2^+ a_2\right) S_-\right] & \text{case } (a) \\ \left[g\rho\left(a_1^+ a_1, a_2^+ a_2\right) a_1^p a_2^p S_+ + g^* a_1^{+p} a_2^{+p} \rho\left(a_1^+ a_1, a_2^+ a_2\right) S_-\right] & \text{case } (b), \end{cases} \tag{47}$$

to which we could add an operator-valued function $\rho_0\left(a_1^+ a_1, a_2^+ a_2\right)$. Case (a) may be associated with a two-mode Raman coupled model with possible $2p$-photon transitions [44]. Case (b), usually treated with $p = 1$ [45–47], may be seen as a QED model in which a two-level atom interacts with two field modes via a non-degenerate $2p$-photon process. In both cases, the fourth and fifth terms describe intensity-dependent Stark shifts commonly taken with $\rho_i\left(a_i^+ a_i\right) = a_i^+ a_i = N_i$. The interaction term involves powers in the ladder operators for $su(2)$ (resp. $su(1,1)$) in its Schwinger [1] (resp. two-mode [48]) realization:

$$\begin{array}{llll} su(2) & J_+ = a_1^+ a_2 & J_- = a_2^+ a_1 & J_z = \frac{1}{2}(N_1 - N_2), \\ su(1,1) & K_+ = a_1^+ a_2^+ & K_- = a_1 a_2 & K_z = \frac{1}{2}(N_1 + N_2 + 1), \end{array} \tag{48}$$

with Casimir invariants, respectively, given by

$$\begin{array}{l} J^2 = [(N_1 + N_2)/2][(N_1 + N_2)/2 + 1)] = (N/2)(N/2 + 1), \\ K^2 = [(N_1 - N_2 + 1)/2][(N_1 - N_2 - 1)/2] = S(S - 1). \end{array} \tag{49}$$

We can thus introduce the conserved quantities

$$\begin{array}{lll} & \text{case } (a) & \text{case } (b) \\ \Delta & J_z + p\left(\frac{1}{2} + S_z\right) & K_z + p\left(\frac{1}{2} + S_z\right), \\ \mathcal{I} & N = N_1 + N_2 & S = \frac{1}{2}(N_1 - N_2 + 1), \end{array} \tag{50}$$

which, together with $\mathcal{F}$ ((29), (31)) and (25), allow us to rewrite (47) in the form (26).

Case (a). For this Raman-type model, $N = N_1 + N_2$ represents the photon total number operator in the pump and Stokes fields with frequency ω_1 and ω_2, respectively, and $J_z = (N_1 - N_2)/2$ the photon difference number operator between the two fields. The latter can thus be seen as a field angular momentum as for doubly degenerate vibrational modes [49, 50]. The field Fock states can be labelled with the quantum numbers associated with the subduction $u(2) \supset su(2) \supset so(2)$ or with the usual eigenvalues of the photon number operators:

$$|[n0]jm\rangle \equiv |n_1, n_2\rangle, \qquad j = \frac{n}{2} = \frac{n_1 + n_2}{2}, \qquad m = \frac{n_1 - n_2}{2}. \tag{51}$$

With $\mathcal{F}$ given from equation (29) by

$$\begin{aligned}\mathcal{F} &= \left(\tfrac{1}{2}+S_z\right)\rho^2(N/2+J_z, N/2-J_z)(N/2+J_z+p)^{\underline{p}}(N/2-J_z)^{\underline{p}}\\ &\quad+\left(\tfrac{1}{2}-S_z\right)\rho^2(N/2+J_z-p, N/2-J_z+p)(N/2+J_z)^{\underline{p}}(N/2-J_z+p)^{\underline{p}}\\ &= \left(\tfrac{1}{2}+S_z\right)\rho^2(N_1,N_2)(N_1+p)^{\underline{p}}N_2^{\underline{p}}+\left(\tfrac{1}{2}-S_z\right)\rho^2(N_1-p,N_2+p)N_1^{\underline{p}}(N_2+p)^{\underline{p}},\end{aligned} \tag{52}$$

we obtain equation (26) with

$$\begin{aligned}H_0 &= (\omega_1+\omega_2)(N_1+N_2)/2,\\ H'(\Delta) &= \left[(\omega_1-\omega_2)\left(\Delta-\frac{p}{2}\right)+\frac{\beta_1}{2}\rho_1\left(\frac{N}{2}+\Delta\right)+\frac{\beta_2}{2}\rho_2\left(\frac{N}{2}-\Delta+p\right),\right.\\ \delta_z(\Delta) &= \omega_0-p(\omega_1-\omega_2)-\beta_1\rho_1\left(\frac{N}{2}+\Delta\right)+\beta_2\rho_2\left(\frac{N}{2}-\Delta+p\right), \qquad \delta=g.\end{aligned} \tag{53}$$

The energies, together with the off-resonance Rabi operator, are determined from equations (30), (37), (38):

$$\begin{aligned}E_{jm\pm} &= \hbar\left\{(\omega_1+\omega_2)j+(\omega_1-\omega_2)\left(m+\frac{p}{2}\right)+\frac{\beta_1}{2}\rho_1(j+m+p)+\frac{\beta_2}{2}\rho_2(j-m)\right.\\ &\quad\left.\pm\left[|g|^2\rho^2(j+m,j-m)\frac{(j-m)!(j+m+p)!}{(j+m)!(j-m-p)!}+\delta_z^2(m+p)/4\right]^{1/2}\right\},\\ \delta_z(m+p) &= \omega_0-p(\omega_1-\omega_2)-\beta_1\rho_1(j+m+p)+\beta_2\rho_2(j-m),\end{aligned} \tag{54}$$

which can alternatively be expressed in terms of the photon numbers n_1, n_2. With equations (17)–(20) various expressions for the eigenstates may be obtained from the initially degenerate states $\Psi_+ = |jm\rangle|+\rangle$ and $\Psi_- = |jm+p\rangle|-\rangle$, $-j \leqslant m \leqslant j-p$ $(j \geqslant p/2)$. The uncoupled states are $|jm\rangle|+\rangle$ $(j-p < m \leqslant j)$ and $|jm\rangle|-\rangle$ $(-j \leqslant m < -j+p)$ with energies, respectively, given by

$$\begin{aligned}E_+ &= \hbar\left[(\omega_1+\omega_2)j+(\omega_1-\omega_2)m+\frac{\omega_0}{2}+\beta_2\rho_2(j-m)\right],\\ E_- &= \hbar\left[(\omega_1+\omega_2)j+(\omega_1-\omega_2)m-\frac{\omega_0}{2}+\beta_1\rho_1(j+m)\right].\end{aligned}$$

The usual version for this model [44], with or without Stark shifts, is recovered with $p = 1$ and $\rho(N_1, N_2) = I$ in the preceding equations.

Case (*b*). Similarly with (25), (50) we can write the effective Hamiltonian (47) in the form of equation (26) with

$$\begin{aligned}H_0 &= (\omega_1-\omega_2)(S-1/2),\\ H'(\Delta) &= (\omega_1+\omega_2)\left(\Delta-\frac{p+1}{2}\right)+\frac{\beta_1}{2}\rho_1(S-1+\Delta)+\frac{\beta_2}{2}\rho_2(-S+\Delta-p),\\ \delta_z(\Delta) &= \omega_0-p(\omega_1+\omega_2)-\beta_1\rho_1(S-1+\Delta)+\beta_2\rho_2(-S+\Delta-p), \qquad \delta=g,\end{aligned} \tag{55}$$

and with $\mathcal{F}$ given from equation (29) by

$$\begin{aligned}\mathcal{F} &= \left(\tfrac{1}{2}+S_z\right)\rho^2(S-1+K_z,-S+K_z)(S-1+K_z+p)^{\underline{p}}(-S+K_z+p)^{\underline{p}}\\ &\quad+\left(\tfrac{1}{2}-S_z\right)\rho^2(S-1+K_z-p,-S+K_z-p)(S-1+K_z)^{\underline{p}}(-S+K_z)^{\underline{p}},\\ &= \left(\tfrac{1}{2}+S_z\right)\rho^2(N_1,N_2)(N_1+p)^{\underline{p}}(N_2+p)^{\underline{p}}+\left(\tfrac{1}{2}-S_z\right)\rho^2(N_1-p,N_2-p)N_1^{\underline{p}}N_2^{\underline{p}}.\end{aligned} \tag{56}$$

With equation (37) this determines the $2p$-photon Rabi operator $\Omega(\mathcal{F}, \Delta)$. The field Fock states $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$ span two equivalent IRs of $su(1,1)$ with bases

$$\begin{aligned} |km\rangle_1 &\equiv |n_1, n_2\rangle, \qquad n_1 = m + 2k - 1, \qquad n_2 = m, \\ |km\rangle_2 &\equiv |n_1, n_2\rangle, \qquad n_1 = m, \qquad n_2 = m + 2k - 1, \end{aligned} \tag{57}$$

except for $n_1 = n_2 = n$, in which case $\left|\frac{1}{2}m = n\right\rangle_1 \equiv \left|\frac{1}{2}m = n\right\rangle_2 \equiv |n_1 = n, n_2 = n\rangle$. The undressed states are $|\Psi_+\rangle = |n_1, n_2\rangle|+\rangle$ and $|\Psi_-\rangle = |n_1 + p, n_2 + p\rangle|-\rangle$. The eigenvalues are obtained with equations (37), (38) with

$$\begin{aligned} f(n_1, n_2) &= \rho^2(n_1, n_2)\frac{(n_1 + p)!(n_2 + p)!}{n_1!n_2!}, \\ \delta_z(n_1, n_2) &= \omega_0 - p(\omega_1 + \omega_2) - \beta_1\rho_1(n_1 + p) + \beta_2\rho_2(n_2), \end{aligned} \tag{58}$$

$$\begin{aligned} E_{n_1 n_2 \pm} = \hbar\Bigg\{&(\omega_1 - \omega_2)(n_1 - n_2)/2 + (\omega_1 + \omega_2)(n_1 + n_2 + p)/2 + \frac{\beta_1}{2}\rho_1(n_1 + p) \\ &+ \frac{\beta_2}{2}\rho_2(n_2) \pm \left\{|g|^2\rho^2(n_1, n_2)\frac{(n_1 + p)!(n_2 + p)!}{n_1!n_2!} + \delta_z^2(n_1, n_2)/4\right\}^{1/2}\Bigg\}, \end{aligned} \tag{59}$$

and may also be expressed in terms of the quantum numbers k, m with (57). It seems to us that some results in [47] should be corrected. With equations (17)–(20) the eigenstates are obtained using the expressions of $\delta_z(\Delta)$ and $\mathcal{F}$ given above. The uncoupled states are $|n_1, n_2\rangle|-\rangle$, $n_1 < p$ or $n_2 < p$, with energies given by $E_{n_1 n_2 -} = \hbar[\omega_1 n_1 + \omega_2 n_2 + \beta_1\rho_1(n_1) - \omega_0/2]$.

We used in (47) the standard two-mode realizations of $su(2)$ and $su(1,1)$. More generally our approach applies to Hamiltonian models:

$$H/\hbar = H_0(R^2, R_z) + \gamma_z(R^2, R_z)S_z + g\rho(R^2, R_z)R_-^p S_+ + g^* R_+^p \rho(R^2, R_z)S_-,$$

or

$$H/\hbar = H_0(R^2, R_z) + \gamma_z(R^2, R_z)S_z + gR_+^p\rho(R^2, R_z)S_+ + g^*\rho(R^2, R_z)R_-^p S_-,$$

for other multi-mode realizations of the $R = J$ or K generators [51–53].

3.2.2. Three-mode nonlinear Raman coupled model. For this Raman model originally proposed in [54], the eigenvalues have been obtained in [55] and it has been recently treated through supersymmetric quantum mechanics [56]. We show below that our formalism gives a complete solution even when the Stark shift and frequency detuning are taken into account. The effective Hamiltonian for this QED model is [55]

$$\begin{aligned} H/\hbar &= \omega_1(N_1 + N_2 + N_3) + E_{+-}\left(a_3^+a_3 - a_2^+a_2 + S_z\right) \\ &\qquad + \left[g\left(a_3^+a_3 - a_2^+a_2\right) + \delta\right](1/2 - S_z) + H_{\text{int}}/\hbar, \\ H_{\text{int}}/\hbar &= g\left[\left(a_2^+a_1 + a_1^+a_3\right)S_+ + \left(a_1^+a_2 + a_3^+a_1\right)S_-\right], \end{aligned} \tag{60}$$

where the indices $i = 1, 2, 3$ refer to the pump, Stokes and anti-Stokes field modes, respectively. The third term contains the Stark shift $g\left(a_3^+a_3 - a_2^+a_2\right)(1/2 - S_z)$ and frequency detuning $\delta(1/2 - S_z)$. Written in the form (60) H appears as a function of the generators of an $u(3)$ algebra with linear invariant $N = N_1 + N_2 + N_3$ associated with the photon total number operator. Among the possible $su(3)$ subalgebras there is $so(3)$ introduced by Wang *et al* [54], which allows us to associate a pseudo-angular momentum $\vec{L}$ with the field modes:

$$L_+ = \sqrt{2}\left(a_1^+a_2 + a_3^+a_1\right), \qquad L_- = \sqrt{2}\left(a_2^+a_1 + a_1^+a_3\right), \qquad L_z = a_3^+a_3 - a_2^+a_2. \tag{61}$$

The well-known properties of the chain $u(3) \supset su(3) \supset so(3) \supset so(2)$ in its three-boson realization [1], not used in [55], together with the isomorphism $so(3) \approx su(2)$ allow us to use our previous results. In particular, the field states are characterized by three quantum numbers n, ℓ, m associated with the IRs of each element in the chain

$$\begin{array}{ccccc} su(3) & \supset & so(3) & \supset & so(2) \\ [n\dot{0}^2] & & \ell & & m \end{array}$$

with $\ell = n, n-2, \ldots, 1$ or $0, m = -\ell, -\ell+1, \ldots, \ell-1, \ell$ just as for the usual three-dimensional isotropic harmonic oscillator.

With $\Delta = L_z + (1/2 + S_z)$ we can rewrite (60) as an '$su(2)$ model' with $p = 1$ (equation (26), table 1):

$$H/\hbar = \omega_1 N - (E_{+-} - \delta)/2 + (E_{+-} + g/2)\Delta - (g\Delta + \delta)S_z + (g/\sqrt{2})(L_- S_+ + L_+ S_-), \quad (62)$$

from which $\widetilde{H}/\hbar$ is obtained in the form (36) with

$$\begin{aligned} &\delta_z(\Delta) = -(g\Delta + \delta), \qquad \mathcal{F} = L^2 - L_z^2 - 2L_z S_z, \\ &\Omega(\mathcal{F}, \Delta) = \tfrac{1}{2}\{2g^2\mathcal{F} + (g\Delta + \delta)^2\}^{1/2}, \end{aligned} \quad (63)$$

and with eigenvalues (equations (30), (38))

$$\begin{aligned} E_{n\ell m\pm} = \hbar \Bigg\{ &\omega_1 n - \frac{1}{2}(E_{+-} - \delta) + \left(E_{+-} + \frac{g}{2}\right)(m+1) \\ &\pm \frac{1}{2}\{2g^2(\ell - m)(\ell + m + 1) + [g(m+1) + \delta]^2\}^{1/2} \Bigg\}. \end{aligned} \quad (64)$$

As before, with equations (17)–(20) and (63) various expressions for the eigenstates may be obtained from the initially degenerates states $|\Psi_+\rangle = |[n\dot{0}^2]\ell m\rangle|+\rangle$ and $|\Psi_-\rangle = |[n\dot{0}^2]\ell m+1\rangle|-\rangle, -\ell \leqslant m \leqslant \ell - 1$. The uncoupled atom–field states are $|\Psi_+\rangle = |[n\dot{0}^2]\ell\ell\rangle|+\rangle$ and $|\Psi_-\rangle = |[n\dot{0}^2]\ell - \ell\rangle|-\rangle$ with energies $E_+ = \hbar[\omega_1 n + E_{+-}(\ell + 1/2)]$ and $E_- = \hbar[\omega_1 n - E_{+-}(\ell + 1/2) - g\ell + \delta]$, respectively. We note that a more convenient expression than those in [55, 56] for the field states in terms of the Fock states $|n_1, n_2, n_3\rangle$ is given by [57, 58]

$$|[n\dot{0}^2]\ell m\rangle = \mathcal{N}(n, \ell, m)\left(a_1^{+2} - 2a_2^+ a_3^+\right)^{(n-\ell)/2} \sum_x \frac{a_1^{+\ell+m-2x} a_2^{+x-m} a_3^{+x}}{2^x x!(x-m)!(\ell+m-2x)!}|0, 0, 0\rangle, \quad (65)$$

with

$$\mathcal{N}(n, \ell, m) = \left[\frac{2^{\ell+m}\left(\frac{n+\ell}{2}\right)!(\ell-m)!(\ell+m)!(2\ell+1)}{\left(\frac{n-\ell}{2}\right)!(n+\ell+1)!}\right]^{1/2}. \quad (66)$$

3.3. The modified two-mode JCM

Among the extensions of JCMs, the modified two-mode JCM describes a two-level atom placed in the common domain of two cavities with equal frequencies [16]. Several techniques have been proposed for its resolution: polynomial expansions [16], algebraic methods [17] and lately via the superalgebra $osp(2, 1)$ [18]. We show below how our formalism gives a simple and compact solution which allows also the extension of the model to an arbitrary number of modes. The Hamiltonian may be written as

$$\begin{aligned} H/\hbar &= \omega\left(a_1^+ a_1 + a_2^+ a_2\right) + \omega_0 S_z + \left(\delta_1 a_1 S_+ + \delta_1^* a_1^+ S_-\right) + \left(\delta_2 a_2 S_+ + \delta_2^* a_2^+ S_-\right) \\ &= \omega\left(a_1^+ a_1 + a_2^+ a_2\right) + \omega_0 S_z + (\delta_1 a_1 + \delta_2 a_2) S_+ + \left(\delta_1^* a_1^+ + \delta_2^* a_2^+\right) S_-, \end{aligned} \quad (67)$$

in which we allow for complex coupling constants. Setting $\delta_k = |\delta_k|\, e^{i\varphi_k} k = 1, 2$ we introduce the unitary transformation U_1 of the annihilation operators a_i $(i = 1, 2)$

$$\begin{aligned} b_1 &= (|\delta_1|^2 + |\delta_2|^2)^{-1/2}(|\delta_1| a_1 + \delta_2\, e^{-i\varphi_1} a_2), \\ b_2 &= (|\delta_1|^2 + |\delta_2|^2)^{-1/2}(-\delta_2^*\, e^{i\varphi_1} a_1 + |\delta_1| a_2), \end{aligned} \quad (68)$$

and similar expressions for the $b_i^+ (i = 1, 2)$ obtained through hermitian conjugation from (68). The commutation relations of the elementary boson operators being preserved the set $b_i^+ b_j$ $(i, j = 1, 2)$ span a $u(2)$ algebra equivalent to that spanned by the $a_i^+ a_j$. In terms of the new boson operators it is easily checked that we have

$$\begin{aligned} H/\hbar &= \omega\big(b_1^+ b_1 + b_2^+ b_2\big) + \omega_0 S_z + (|\delta_1|^2 + |\delta_2|^2)^{1/2}\big(\mathrm{e}^{\mathrm{i}\varphi_1} b_1 S_+ + \mathrm{e}^{-\mathrm{i}\varphi_1} b_1^+ S_-\big) \\ &= \omega b_2^+ b_2 + \omega b_1^+ b_1 + \omega_0 S_z + \big(\delta_1' b_1 S_+ + \delta_1'^* b_1^+ S_-\big), \end{aligned} \tag{69}$$

which is the sum of two uncoupled Hamiltonians—one associated with a one-dimensional harmonic oscillator and the other with a one-photon JCM. With the results of section 3.1 a second unitary transformation leads to

$$U_2 (H/\hbar) U_2^{-1} = \widetilde{H}/\hbar = \omega b_2^+ b_2 + \omega\big(\Delta - \tfrac{1}{2}\big) + 2\Omega(\Delta) S_z, \tag{70}$$

where

$$\Delta = \mathcal{F} = b_1^+ b_1 + 1/2 + S_z, \qquad \Omega(\Delta) = \left[(|\delta_1|^2 + |\delta_2|^2)\Delta + \frac{(\omega_0 - \omega)^2}{4}\right]^{1/2}.$$

In the basis $|\widehat{n_1, n_2}\rangle|+\rangle$, $|\widehat{n_1 + 1, n_2}\rangle|-\rangle$, where $|\widehat{n_1, n_2}\rangle$ is the eigenbasis of the new number operators $N_i = b_i^+ b_i$, we thus obtain the eigenvalues of (70) and (67):

$$E(n_1, n_2, \pm) = \hbar\omega n_2 + \hbar\omega\left(n_1 + \frac{1}{2}\right) \pm \hbar\left[(|\delta_1|^2 + |\delta_2|^2)(n_1 + 1) + \frac{(\omega_0 - \omega)^2}{4}\right]^{1/2}. \tag{71}$$

With the results in the appendix the states $|\widehat{n_1, n_2}\rangle$ may be written in several forms:

$$\begin{aligned} |\widehat{n_1, n_2}\rangle &= U_1 |n_1, n_2\rangle = (n_1! n_2!)^{-1/2} b_1^{+n_1} b_2^{+n_2} |0, 0\rangle \\ &\equiv |\widehat{jm}\rangle = U_1 |jm\rangle = \exp[\xi J_- - \xi^* J_+] |jm\rangle, \end{aligned} \tag{72}$$

with j and m as defined in equation (51) and

$$\xi = |\xi|\, \mathrm{e}^{\mathrm{i}(\varphi_1 - \varphi_2)}, \qquad \tan|\xi| = |\delta_2|/|\delta_1|. \tag{73}$$

The second form in (72) shows more clearly the $su(2)$ symmetry; the corresponding Euler–Rodrigues parameters are (3)

$$\begin{aligned} \lambda &= \cos\frac{\varphi}{2} = \frac{|\delta_1|}{(|\delta_1|^2 + |\delta_2|^2)^{1/2}}, \\ \vec{\Lambda} &= \sin\frac{\varphi}{2}\vec{n} = \frac{|\delta_2|}{(|\delta_1|^2 + |\delta_2|^2)^{1/2}} (\sin(\varphi_1 - \varphi_2), -\cos(\varphi_1 - \varphi_2), 0). \end{aligned} \tag{74}$$

The energies (71) can alternatively be expressed in terms of the quantum numbers associated with the total photon number $n = n_1 + n_2$ and n_1 or $j = n/2$ and m.

The dressed states of H (67) are given by

$$|\widetilde{\Psi}_\pm\rangle = U_2^{-1} |\widehat{jm}\rangle|\pm\rangle = U_2^{-1} U_1 |jm\rangle|\pm\rangle, \tag{75}$$

where the U_2^{-1} transformation is that of equations (17), (18) with

$$k_- = -\frac{1}{\delta_1' \sqrt{\Delta}} [\Omega(\Delta) - (\omega_0 - \omega)/2], \qquad k_z = \ln\left[\frac{2\Omega(\Delta)}{\Omega(\Delta) + (\omega_0 - \omega)/2}\right].$$

Alternatively they may be taken in the form of equation (46) with the substitutions

$$\begin{aligned} &\Omega(f) \to \Omega(n_1 + 1), && \delta_z(\kappa) \to (\omega_0 - \omega), \\ &|n\rangle|+\rangle \to |\widehat{n_1, n_2}\rangle|+\rangle, && |n + p\rangle|-\rangle \to |\widehat{n_1 + 1, n_2}\rangle|-\rangle. \end{aligned}$$

These relations together with equations (68), (72)–(74) allow us to obtain compact expressions for the eigenstates in terms of the initial basis $|n_1, n_2\rangle$ associated with modes 1 and 2. Our results are in agreement with those obtained in [17]. Our introduction of a first transformation U_1 acting on the field modes alone simplifies notably the expressions of the eigenstates. In particular, we point out the case of the null dressed states $\widehat{|0, n_2\rangle}|-\rangle$, which are eigenstates of H, as shown by (69), with energies $E_{n_2} = \hbar(\omega n_2 - \omega_0/2)$. With equations (72), (73) one shows that they involve standard $su(2)$ coherent states [59]:

$$\widehat{|0, n_2\rangle}|-\rangle = \widehat{|j - j\rangle}|-\rangle = \left[1 + \frac{|\delta_2|^2}{|\delta_1|^2}\right]^{-j} \exp\left[-\frac{\delta_2}{\delta_1} J_+\right] |j - j\rangle|-\rangle, \qquad j = \frac{n_2}{2}.$$

Alternatively with (68), (72) a very simple expression is obtained in terms of the initial Fock states with q and $n_2 - q$ photons in modes 1 and 2, respectively:

$$\widehat{|0, n_2\rangle}|-\rangle = \left[\frac{|\delta_1|}{(|\delta_1|^2 + |\delta_2|^2)^{1/2}}\right]^{n_2} \sum_{q=0}^{n_2} (-1)^q (\delta_2/\delta_1)^q \binom{n_2}{q}^{1/2} |q n_2 - q\rangle|-\rangle.$$

Our method also suggests a generalization to an arbitrary number of modes which we briefly sketch. We consider the extension of (67) to a p-mode case

$$\begin{aligned} H/\hbar &= \omega \sum_{i=1}^{p} a_i^+ a_i + \omega_0 S_z + \sum_{i=1}^{p} \left(\delta_i a_i S_+ + \delta_i^* a_i^+ S_-\right) \\ &= \omega \sum_{i=1}^{p} a_i^+ a_i + \omega_0 S_z + \left(\sum_{i=1}^{p} \delta_i a_i\right) S_+ + \left(\sum_{i=1}^{p} \delta_i^* a_i^+\right) S_-, \end{aligned} \tag{76}$$

the first term of which involves the linear invariant $N = \sum_{i=1}^{p} a_i^+ a_i$ of a $u(p)$ algebra preserved under any $p \times p$ unitary transformation

$$b_i = \sum_{j=1}^{p} U_i^j a_j, \qquad b_i^+ = \sum_{j=1}^{p} U_i^{j*} a_j^+.$$

One may always choose U such that

$$\begin{aligned} b_1 &= \mathcal{N}^{-1/2} \left[|\delta_1| a_1 + \sum_{i=2}^{p} \delta_i \, \mathrm{e}^{-\mathrm{i}\varphi_1} a_i\right], \qquad \mathcal{N} = \sum_{i=1}^{p} |\delta_i|^2, \\ b_1^+ &= \mathcal{N}^{-1/2} \left[|\delta_1| a_1^+ + \sum_{i=2}^{p} \delta_i^* \, \mathrm{e}^{\mathrm{i}\varphi_1} a_i^+\right], \end{aligned}$$

and the Hamiltonian (76) is written in terms of the new field mode operators

$$\begin{aligned} H/\hbar &= \omega \sum_{i=1}^{p} b_i^+ b_i + \omega_0 S_z + \mathcal{N}^{1/2} \left(\mathrm{e}^{\mathrm{i}\varphi_1} b_1 S_+ + \mathrm{e}^{-\mathrm{i}\varphi_1} b_1^+ S_-\right) \\ &= \omega \sum_{i=2}^{p} b_i^+ b_i + \omega b_1^+ b_1 + \omega_0 S_z + \left(\delta_1' b_1 S_+ + \delta_1'^* b_1^+ S_-\right), \end{aligned} \tag{77}$$

which is of the same form as (69) but with an uncoupled isotropic harmonic oscillator with dimension $p - 1$ instead of 1. The second step in the diagonalization procedure is then identical to that discussed previously. We note that the arbitrariness which remains for the determination of the U transformation can be raised with the choice of a canonical symmetry adaptation [60] for the generators $b_i^+ b_j$ $(i, j = 2, \ldots, p)$ spanning a $u(p - 1)$ algebra.

3.4. $E \otimes \varepsilon$ Jahn–Teller systems

Several problems in molecular spectroscopy [3, 50, 61] involve an $u(2)$ algebra in its Schwinger realization. According to the case, the bosonic variables are associated with different degrees of freedom. For instance, a $u_v(2)$ vibrational algebra is useful for the treatment of doubly degenerate vibrational modes [49]. Also orbital doublets may be described in terms of an electronic $u_e(2)$ algebra [50, 62, 63] which is that of a spin 1/2; in this case, the pseudo-spin components $S_\alpha = \sigma_\alpha/2$ are symmetry adapted electronic operators.

With the results of section 2 we could take a Hamiltonian model in one of the general forms (26) or (35) with realizations of S_+A and $\mathcal{F}$ (or $\overline{\mathcal{F}}$) as given in (22) and (29). It would lead to, in particular, a whole class of '$su(2)$ JCMs'. Although solvable these are not quite realistic for our purpose since zeroth-order approximations in vibronic spectroscopy usually involve low powers in the elementary operators and it is sufficient to consider cases when the power p in A is one and $\rho(J_z) = I$ in (22). More precisely in our first example the dominant interaction is modelled by

$$H_{\text{int}\pm} \propto S_\alpha J_\beta \pm S_\gamma J_\delta, \qquad \alpha \neq \gamma, \quad \beta \neq \delta, \tag{78}$$

where the J_i operators are associated with a doubly degenerate vibrational ε mode [49, 50]. This kind of interaction appears in several $E \otimes \varepsilon$ Jahn–Teller systems [63–65] and the values for the indices $\alpha, \beta, \ldots,$ depend on the specific molecule under consideration. The H_0 term is usually taken as the oscillator Hamiltonian $H_0 = \hbar\omega(N+1)$ but could be more generally an operator valued function of the total number operator N or of $J^2 = (N/2)(N/2+1)$.

For arbitrary value of α, γ ($\alpha \neq \gamma$) and β, δ ($\beta \neq \delta$), we can define

$$\begin{aligned} \widehat{S}_\pm &= S_\alpha \pm \mathrm{i}S_\gamma, & \widehat{J}_\pm &= J_\beta \pm \mathrm{i}J_\delta, \\ \widehat{S}_z &= \epsilon_{\alpha\gamma\theta} S_\theta, & \widehat{J}_z &= \epsilon_{\beta\delta\tau} J_\tau, \end{aligned} \tag{79}$$

which satisfy the usual $su(2)$ commutation rules $[\widehat{W}_z, \widehat{W}_\pm] = \pm\widehat{W}_\pm$, $[\widehat{W}_+, \widehat{W}_-] = 2\widehat{W}_z$ and $\widehat{W}^2 = W^2$ ($\widehat{W} = \widehat{S}$ or $\widehat{J}$). The normal standard basis is replaced by ($w = 1/2$ or j)

$$\begin{aligned} &|\widehat{w, m}\rangle = P_R |w, m\rangle, \\ &\widehat{W}_z |\widehat{w, m}\rangle = m |\widehat{w, m}\rangle, \qquad \widehat{W}^2 |\widehat{w, m}\rangle = w(w+1) |\widehat{w, m}\rangle, \end{aligned} \tag{80}$$

where P_R are rotation operators which perform the changes $(x, y, z) \to (\alpha, \gamma, \theta)$ and $(x, y, z) \to (\beta, \delta, \tau)$. These are given in appendix A.

With (79) the interaction terms (78) are written as

$$H_{\text{int}+} \propto \tfrac{1}{2}(\widehat{S}_+\widehat{J}_- + \widehat{S}_-\widehat{J}_+) = \tfrac{1}{2}\mathcal{O}, \qquad H_{\text{int}-} \propto \tfrac{1}{2}(\widehat{S}_+\widehat{J}_+ + \widehat{S}_-\widehat{J}_-) = \tfrac{1}{2}\overline{\mathcal{O}}, \tag{81}$$

which is in the form of equation (9). Thus for our Hamiltonian models

$$H_\pm/\hbar = \omega(N+1) + 2g(S_\alpha J_\beta \pm S_\gamma J_\delta), \tag{82}$$

($g \in \mathbb{R}$) all results of section 2 apply directly. With notation similar to that of table 1 in which the substitutions $S_\pm \to \widehat{S}_\pm$, $S_z \to \widehat{S}_z$, $J_\pm \to \widehat{J}_\pm$ and $J_z \to \widehat{J}_z$ are made, we have

$$\begin{aligned} H_+/\hbar &= \omega(N+1) + g\sqrt{\mathcal{F}}(\widehat{M}_+ + \widehat{M}_-), \\ H_-/\hbar &= \omega(N+1) + g\sqrt{\overline{\mathcal{F}}}(\widehat{N}_+ + \widehat{N}_-), \end{aligned} \tag{83}$$

which is equation (12) with $\delta_z = 0$. After the unitary transformation U we have (equations (15), (16))

$$\begin{aligned} U(H_+/\hbar)U^{-1} &= \widetilde{H}_+/\hbar = \omega(N+1) + 2g\sqrt{\mathcal{F}}\widehat{S}_z, \\ U(H_-/\hbar)U^{-1} &= \widetilde{H}_-/\hbar = \omega(N+1) + 2g\sqrt{\overline{\mathcal{F}}}\widehat{S}_z, \end{aligned} \tag{84}$$

with eigenvalues ($j = n/2, m : -j, \ldots, j-1$)

$$E(j, m, \pm) = \hbar\{\omega(2j+1) \pm g[(j-m)(j+m+1)]^{1/2}\}, \tag{85}$$

respectively, in the bases

$$\begin{aligned} &|\widehat{j, m}\rangle|\widehat{+}\rangle, |\widehat{j, m+1}\rangle|\widehat{-}\rangle \qquad \text{for } \widetilde{H}_+, \\ &|\widehat{j, m+1}\rangle|\widehat{+}\rangle, |\widehat{j, m}\rangle|\widehat{-}\rangle \qquad \text{for } \widetilde{H}_-, \end{aligned} \tag{86}$$

where $|\widehat{\pm}\rangle = |\widehat{1/2, \pm 1/2}\rangle$ is the electronic basis.

The uncoupled states are $|\widehat{j, \pm j}\rangle|\widehat{\pm}\rangle$ for $\widetilde{H}_+$ (resp. $|\widehat{j, \mp j}\rangle|\widehat{\pm}\rangle$ for $\widetilde{H}_-$) and they are eigenstates of H_+ (resp. H_-) with the same eigenvalue $E = \hbar\omega(2j+1)$.

With equations (19)–(21) the vibronic eigenstates of $H_\pm$ associated with the eigenvalues (85) have the simple form

$$\begin{array}{ll} \qquad\qquad H_+ & \qquad\qquad H_- \\ |^+\widetilde{\Psi}_+\rangle = \frac{1}{\sqrt{2}}[|\widehat{j, m}\rangle|\widehat{+}\rangle + |\widehat{j, m+1}\rangle|\widehat{-}\rangle], & |^-\widetilde{\Psi}_+\rangle = \frac{1}{\sqrt{2}}[|\widehat{j, m+1}\rangle|\widehat{+}\rangle + |\widehat{j, m}\rangle|\widehat{-}\rangle] \\ |^+\widetilde{\Psi}_-\rangle = \frac{1}{\sqrt{2}}[|\widehat{j, m+1}\rangle|\widehat{-}\rangle - |\widehat{j, m}\rangle|\widehat{+}\rangle], & |^-\widetilde{\Psi}_-\rangle = \frac{1}{\sqrt{2}}[|\widehat{j, m}\rangle|\widehat{-}\rangle - |\widehat{j, m+1}\rangle|\widehat{+}\rangle]. \end{array} \tag{87}$$

They can be further expressed in terms of the initial basis with (80) and the results in the appendix for specific values of (α, γ, θ) and (β, δ, τ).

We note that each eigenstate is still doubly degenerate $E(j, m, \pm) = E(j, -m-1, \pm)$. This comes from an initial degeneracy of four of the eigenvalues of the operators $\mathcal{F}$ and $\overline{\mathcal{F}}$ (and $2j+1$ for those of H_0). This remaining degeneracy could be raised by the introduction in the Hamiltonian expansion of a term (equations (23), (33))

$$\omega_0 S_\theta + \lambda_z J_\tau = \omega_0 \widehat{S}_z + \lambda_z \widehat{J}_z = \omega_0 \widehat{S}_z + \lambda_z \begin{cases} (\Delta - \widehat{S}_z) & \text{in } H_+, \\ (\overline{\Delta} + \widehat{S}_z) & \text{in } H_-, \end{cases}$$

with $\lambda_z \neq 0$ in order to raise the degeneracy and the models would be equivalent (at least for H_+) to a one-photon two-mode Raman coupled model (section 3.2.1). However, all these terms are precluded by symmetry and by the time reversal invariance of the vibronic Hamiltonian [63, 66]. More interesting is the introduction of the allowed operator $S_\theta J_\tau = \widehat{S}_z \widehat{J}_z$, which can be associated with an effective pseudo-spin-vibration interaction [65]. The additional interaction term can be written as

$$\lambda_z \widehat{S}_z \widehat{J}_z = \begin{cases} -\lambda_z/4 + \lambda_z \Delta \widehat{S}_z & \text{in } H_+ \\ +\lambda_z/4 + \lambda_z \overline{\Delta} \widehat{S}_z & \text{in } H_-. \end{cases} \tag{88}$$

Thus the vibronic Hamiltonians (83) become

$$\begin{aligned} H'_+/\hbar &= \omega(N+1) - \lambda_z/4 + g\sqrt{\mathcal{F}}(\widehat{M}_+ + \widehat{M}_-) + \lambda_z \Delta \widehat{S}_z \\ H'_-/\hbar &= \omega(N+1) + \lambda_z/4 + g\sqrt{\overline{\mathcal{F}}}(\widehat{N}_+ + \widehat{N}_-) + \lambda_z \overline{\Delta} \widehat{S}_z, \end{aligned} \tag{89}$$

which is in the form of equation (26) for H'_+ ((35) for H'_-) with $\delta_z(\Delta) = \hbar\lambda_z\Delta$ ($\delta_z(\overline{\Delta}) = \hbar\lambda_z\overline{\Delta}$). With equations (36), (38) we obtain

$$\begin{aligned} U(H'_+/\hbar)U^{-1} &= \omega(N+1) - \lambda_z/4 + 2\Omega(\mathcal{F}, \Delta)\widehat{S}_z, \\ U(H'_-/\hbar)U^{-1} &= \omega(N+1) + \lambda_z/4 + 2\overline{\Omega}(\overline{\mathcal{F}}, \overline{\Delta})\widehat{S}_z, \end{aligned} \tag{90}$$

with

$$\Omega(\mathcal{F}, \Delta) = \left[g^2\mathcal{F} + \lambda_z^2\Delta^2/4\right]^{1/2}, \qquad \overline{\Omega}(\overline{\mathcal{F}}, \overline{\Delta}) = \left[g^2\overline{\mathcal{F}} + \lambda_z^2\overline{\Delta}^2/4\right]^{1/2},$$

and the corresponding eigenvalues

$$^{\pm}E'(j,m,\pm) = \hbar\omega(2j+1) \mp \hbar\lambda_z/4 \pm \hbar\left[g^2(j-m)(j+m+1) + \frac{\lambda_z^2}{4}\left(m+\frac{1}{2}\right)^2\right]^{1/2}, \tag{91}$$

associated with the same states than in equation (86). The eigenstates of $H'_{\pm}$ are obtained with the unitary transformations (equations (17), (18)),

$$^{+}U^{-1} = \exp[-(\xi M_+ - \xi^\dagger M_-)], \qquad ^{-}U^{-1} = \exp[-(\bar{\xi} N_+ - \bar{\xi}^\dagger N_-)], \tag{92}$$

acting on these same states. Setting $g = |g|\,\mathrm{e}^{\mathrm{i}\varphi}$ ($\varphi = 0$ or π) we have

$$\xi = |\xi|\,\mathrm{e}^{\mathrm{i}\varphi}, \tan|\xi| = \left[\frac{\Omega(\mathcal{F},\Delta) - \lambda_z\Delta/2}{\Omega(\mathcal{F},\Delta) + \lambda_z\Delta/2}\right]^{1/2},$$

and $\bar{\xi}$ with the substitutions $\Omega \to \overline{\Omega}$ and $\Delta \to \overline{\Delta}$. Explicitly this gives:

- For H'_+

$$|^{+}\widetilde{\Psi}_+\rangle = {}^{+}U^{-1}|\widehat{j,m}\rangle|\widehat{+}\rangle = \cos\theta(j,m)|\widehat{j,m}\rangle|\widehat{+}\rangle + \mathrm{e}^{-\mathrm{i}\varphi}\sin\theta(j,m)|\widehat{j,m+1}\rangle|\widehat{-}\rangle,$$
$$|^{+}\widetilde{\Psi}_-\rangle = {}^{+}U^{-1}|\widehat{j,m+1}\rangle|\widehat{-}\rangle = \cos\theta(j,m)|\widehat{j,m+1}\rangle|\widehat{-}\rangle - \mathrm{e}^{\mathrm{i}\varphi}\sin\theta(j,m)|\widehat{j,m}\rangle|\widehat{+}\rangle. \tag{93}$$

- For H'_-

$$|^{-}\widetilde{\Psi}_+\rangle = {}^{-}U^{-1}|\widehat{j,m+1}\rangle|\widehat{+}\rangle = \cos\theta(j,m)|\widehat{j,m+1}\rangle|\widehat{+}\rangle + \mathrm{e}^{-\mathrm{i}\varphi}\sin\theta(j,m)|\widehat{j,m}\rangle|\widehat{-}\rangle,$$
$$|^{-}\widetilde{\Psi}_-\rangle = {}^{-}U^{-1}|\widehat{j,m}\rangle|\widehat{-}\rangle = \cos\theta(j,m)|\widehat{j,m}\rangle|\widehat{-}\rangle - \mathrm{e}^{\mathrm{i}\varphi}\sin\theta(j,m)|\widehat{j,m+1}\rangle|\widehat{+}\rangle, \tag{94}$$

with

$$\begin{aligned}
\cos\theta(j,m) &= \left[\frac{\Omega(j,m) + \lambda_z\Delta(j,m)/2}{2\Omega(j,m)}\right]^{1/2},\\
\sin\theta(j,m) &= \left[\frac{\Omega(j,m) - \lambda_z\Delta(j,m)/2}{2\Omega(j,m)}\right]^{1/2},\\
\Omega(j,m) &= \left[g^2(j-m)(j+m+1) + \lambda_z^2(m+1/2)^2/4\right]^{1/2},\\
\Delta(j,m) &= (m+1/2).
\end{aligned} \tag{95}$$

As for $H_{\pm}$, the expression of the eigenstates in the initial bases are obtained with (80) and the results in the appendix. We will not discuss here the additional step required for the obtention of vibronic eigenstates symmetrized in the molecular point group G.

As an aside we note that in the special case where $g = \lambda_z/2$ in H'_+ it reduces to

$$H''_+/\hbar = \omega(N+1) + \lambda_z\vec{\hat{S}}\cdot\vec{\hat{J}} = \omega(N+1) + \frac{\lambda_z}{2}[(\vec{\hat{S}}+\vec{\hat{J}})^2 - \vec{\hat{J}}^2 - 3/4], \tag{96}$$

the eigensolutions of which are well known:

$$^{+}E(j',j) = \hbar\left\{\omega(2j+1) + \frac{\lambda_z}{2}[j'(j'+1) - j(j+1) - 3/4]\right\}, \qquad j' = j \pm 1/2,$$
$$|j,\widehat{1/2;j'm'}\rangle = \sum_{m,m_e} C\begin{matrix} m & m_e & (j') \\ (j & \frac{1}{2}) & m' \end{matrix}\,|\widehat{j,m}\rangle|\widehat{1/2,m_e}\rangle. \tag{97}$$

$|j, \widehat{1/2; j'}m'\rangle$ is the coupled basis associated with $\vec{\vec{J'}} = \vec{\vec{S}} + \vec{\vec{J}}$ and the C coefficients are the usual $su(2)$ Clebsch–Gordan coefficients [67]. Each energy level has a degeneracy of $2j'+1$. It can easily be checked that, when $g = \lambda_z/2$, the eigenvalues in (91) reduce to those in (97). Also from (91), (93), (95) the correlation between eigenstates reads

$$\begin{aligned} &\text{for } \lambda_z > 0 \quad |^{+}\widetilde{\Psi}_{+}\rangle \equiv \left|j, \tfrac{1}{2}; \widehat{j+\tfrac{1}{2}}m'\right\rangle, \qquad |^{+}\widetilde{\Psi}_{-}\rangle \equiv \left|j, \tfrac{1}{2}; \widehat{j-\tfrac{1}{2}}m'\right\rangle, \\ &\text{for } \lambda_z < 0 \quad |^{+}\widetilde{\Psi}_{+}\rangle \equiv \left|j, \tfrac{1}{2}; \widehat{j-\tfrac{1}{2}}m'\right\rangle, \qquad |^{+}\widetilde{\Psi}_{-}\rangle \equiv \left|j, \tfrac{1}{2}; \widehat{j+\tfrac{1}{2}}m'\right\rangle; \end{aligned}$$

in other words, in this case ($g = \lambda_z/2$),

$$\cos\theta(j,m) = \left[\frac{j+m+1}{2j+1}\right]^{\frac{1}{2}} = C\begin{matrix} m & \frac{1}{2} & (j+\frac{1}{2}) \\ (j & \frac{1}{2}) & m+\frac{1}{2} \end{matrix} = C\begin{matrix} m+1 & -\frac{1}{2} & (j-\frac{1}{2}) \\ (j & \frac{1}{2}) & m+\frac{1}{2} \end{matrix},$$

$$\sin\theta(j,m) = \left[\frac{j-m}{2j+1}\right]^{\frac{1}{2}} = C\begin{matrix} m+1 & -\frac{1}{2} & (j+\frac{1}{2}) \\ (j & \frac{1}{2}) & m+\frac{1}{2} \end{matrix} = -C\begin{matrix} m & \frac{1}{2} & (j-\frac{1}{2}) \\ (j & \frac{1}{2}) & m+\frac{1}{2} \end{matrix}.$$

4. Further extensions

In section 2 we assumed A operators involving powers in one of the ladder operator of $\mathcal{A}$. However, there are circumstances in which pseudo-spin interactions $S_+A + S_-A^\dagger$ involve more general A operators for which equations (2),(5)–(7) are valid. The main step is thus to solve the eigenvalue equation for $\mathcal{F}$ (2). Next solvable models can be built with appropriate additional terms corresponding to $H_0 + H'$ in equations (26), (35).

We illustrate these situations below through two examples—one in which no diagonalization of the $\mathcal{F}$ operator is required; the other, and more interesting, in which $\mathcal{F}$ is a function in the generators of a new algebra $\mathcal{B}$.

4.1. Non-degenerate two-mode multiquanta JCM

We consider the Hamiltonian models with $p_1 \neq p_2$:

$$\begin{aligned} H/\hbar &= \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \omega_0 S_z + H_{\text{int}}/\hbar, \\ H_{\text{int}}/\hbar &= \begin{cases} g\left[\rho(N_1,N_2) a_2^{+p_2} a_1^{p_1} S_+ + a_1^{+p_1} a_2^{p_2} \rho(N_1,N_2) S_-\right] & \text{case (a)}, \\ g\left[\rho(N_1,N_2) a_1^{p_1} a_2^{p_2} S_+ + a_1^{+p_1} a_2^{+p_2} \rho(N_1,N_2) S_-\right] & \text{case (b)}. \end{cases} \end{aligned} \tag{98}$$

They can be seen as generalizations of those considered in sections 3.2.1; to simplify we omit the equivalent of the Stark shift terms of equation (47). They are also effective Hamiltonians for cavity QED with cold trapped ions [68, 69] when the trapping potential is modelled by a two-dimensional harmonic oscillator and the electromagnetic field treated classically. The boson operators are then associated with the quantized vibrational motion of the ion and p_i is the number of quanta in mode i. The appropriate model (a) or (b) is determined by the tuning of the laser frequency to a specific vibrational sideband [69]. Model (b) has been used recently [70] to investigate the influence of the intrinsic decoherence on non-classical effects.

As $p_1 \neq p_2$ H_{int} is no longer a function of the $su(2)$ (resp. $su(1,1)$) generators and $[H, N_1+N_2] \neq 0$ (resp. $[H, N_1-N_2] \neq 0$). However, since we have, from equation (2),

$$\begin{aligned} \mathcal{F} &= \left(\tfrac{1}{2}+S_z\right)\rho^2(N_1,N_2)(N_1+p_1)^{\underline{p_1}} N_2^{\underline{p_2}} && \text{case (a)} \\ &\quad + \left(\tfrac{1}{2}-S_z\right)\rho^2(N_1-p_1,N_2+p_2) N_1^{\underline{p_1}} (N_2+p_2)^{\underline{p_2}}, \\ \mathcal{F} &= \left(\tfrac{1}{2}+S_z\right)\rho^2(N_1,N_2)(N_1+p_1)^{\underline{p_1}} (N_2+p_2)^{\underline{p_2}} && \text{case (b)} \\ &\quad + \left(\tfrac{1}{2}-S_z\right)\rho^2(N_1-p_1,N_2-p_2) N_1^{\underline{p_1}} N_2^{\underline{p_2}}, \end{aligned} \tag{99}$$

the degenerate states of $\mathcal{F}$ are

$$\begin{array}{llll} |\Psi_+\rangle & |\Psi_-\rangle & & \\ |[n0]jm\rangle|+\rangle \equiv |n_1, n_2+p_2\rangle|+\rangle, & |[n'0]j'm'\rangle|-\rangle \equiv |n_1+p_1, n_2\rangle|-\rangle & \text{case (a)}, & (100) \\ |km\rangle|+\rangle \equiv |n_1, n_2\rangle|+\rangle, & |k'm'\rangle|-\rangle \equiv |n_1+p_1, n_2+p_2\rangle|-\rangle & \text{case (b)}, & \end{array}$$

the form of which (equations (51), (57)) shows clearly the broken $su(2)$ and $su(1,1)$ symmetries for cases (a) and (b), respectively. The associated eigenvalues $f(n_1, n_2)$ are

$$f(n_1, n_2) = \frac{(n_1+p_1)!(n_2+p_2)!}{n_1!n_2!} \times \begin{cases} \rho^2(n_1, n_2+p_2) & \text{case (a)}, \\ \rho^2(n_1, n_2) & \text{case (b)}. \end{cases} \tag{101}$$

Also it is easily checked that the two commuting operators[2],

$$\begin{array}{lll} \Delta_1 = N_1 + p_1\left(\frac{1}{2} + S_z\right) & \Delta_2 = N_2 + p_2\left(\frac{1}{2} - S_z\right) & \text{case (a)}, \\ \Delta_1 = N_1 + p_1\left(\frac{1}{2} + S_z\right) & \Delta_2 = N_2 + p_2\left(\frac{1}{2} + S_z\right) & \text{case (b)}, \end{array} \tag{102}$$

commute with the $su(2)(P)$ generators as defined in (6) and satisfy in both cases

$$\Delta_i|\Psi_\pm\rangle = (n_i + p_i)|\Psi_\pm\rangle \qquad i = 1, 2. \tag{103}$$

Thus the Hamiltonians (98) can be written in a form similar to (26)

$$H/\hbar = \omega_1(\Delta_1 - p_1/2) + \omega_2(\Delta_2 - p_2/2) + \delta_z P_z + g\sqrt{\mathcal{F}}(P_+ + P_-),$$

with the detuning parameters $\delta_z = \omega_0 - p_1\omega_1 + p_2\omega_2$ (resp. $\delta_z = \omega_0 - p_1\omega_1 - p_2\omega_2$) for case (a) (resp. for case (b)) and where $H' = \omega_1(\Delta_1 - p_1/2) + \omega_2(\Delta_2 - p_2/2)$ is invariant under any $su(2)(P)$ transformation. Thus $\widetilde{H}/\hbar = H' + 2\Omega(\mathcal{F})S_z$ with $\Omega(\mathcal{F}) = \left[g^2\mathcal{F} + \delta_z^2/4\right]^{1/2}$. The eigenstates of H are obtained directly from equations (17)–(20). The equivalent of the null dressed states are given below together with the corresponding eigenvalues:

$$\begin{array}{llll} |n_1, n_2\rangle|+\rangle & n_2 = 0, 1, \ldots, p_2 - 1 & E_+ = \hbar[\omega_1 n_1 + \omega_2 n_2 + \omega_0/2] & \text{case (a)}, \\ |n_1, n_2\rangle|-\rangle & n_1 = 0, 1, \ldots, p_1 - 1 & E_- = \hbar[\omega_1 n_1 + \omega_2 n_2 - \omega_0/2] & \text{case (a)}, \\ |n_1, n_2\rangle|-\rangle & n_1 = 0, 1, \ldots, p_1 - 1 & E_- = \hbar[\omega_1 n_1 + \omega_2 n_2 - \omega_0/2] & \text{case (b)}. \\ & n_2 = 0, 1, \ldots, p_2 - 1 & & \end{array}$$

We think that these results should allow a notable simplification of the calculations developed in [70] since the field coherent states can easily be expressed in the eigenbasis $|\widetilde{\Psi}_{n_1 n_2 \pm}\rangle$ of H.

4.2. Two-channel Raman model

This cavity QED model introduced in [71] is of the same non-degenerate type as the one considered in section 3.2.1 (case (a)) but has a classical pump field. The effective Hamiltonian reads

$$H = E_{31}\left(a_1^+a_1 - a_2^+a_2 + S_z\right) + \left(\eta a_1 + \xi a_2^+\right)S_+ + \left(\eta a_1^+ + \xi a_2\right)S_-, \tag{104}$$

where the indices $i = 1, 2$ refer to the anti-Stokes and Stokes fields, respectively; E_{31} is the energy gap between levels one and three with the Raman two-photon resonance condition $\omega_1 - \omega_P = \omega_P - \omega_2 = E_{31}$. The parameters η and ξ characterize the atom–field couplings. The problem has been considered in [71] with the assumption of equal couplings; we treat here the actual situation in which $\eta \neq \xi$.

In (104) the field operators span an $\mathcal{A} = h_4(1) \oplus h_4(2)$ algebra but the $\mathcal{F}$ operator,

$$\begin{aligned} \mathcal{F} &= \left[\left(\eta a_1 + \xi a_2^+\right)S_+ + \left(\eta a_1^+ + \xi a_2\right)S_-\right]^2, \\ &= (\eta^2 - \xi^2)(N_1 - N_2)/2 + (\eta^2 + \xi^2)(N_1 + N_2 + 1)/2 + \eta\xi\left(a_1a_2 + a_1^+a_2^+\right) + (\eta^2 - \xi^2)S_z, \end{aligned} \tag{105}$$

[2] Δ_2 for case (a) is in fact of type $\overline{\Delta}$.

is a linear combination in the generators of a $su(1,1)$ algebra in a two-boson realization. Since $N_1 - N_2$ commutes with all $su(1,1)$ generators a unitary transformation V [19, 20] of the field operators gives

$$V\mathcal{F}V^{-1} = \begin{cases} (\eta^2 - \xi^2)\left(N_1 + \frac{1}{2} + S_z\right) & \eta^2 > \xi^2, \\ (\xi^2 - \eta^2)\left(N_2 + \frac{1}{2} - S_z\right) & \eta^2 < \xi^2, \end{cases} \tag{106}$$

where V is the two-mode squeeze operator

$$V = \exp[f_0(K_- - K_+)] = \exp\left[f_0\left(a_1 a_2 - a_1^+ a_2^+\right)\right], \tag{107}$$

with $\tanh f_0 = \xi^2/\eta^2$ (resp. $\tanh f_0 = \eta^2/\xi^2$) for $\eta^2 > \xi^2$ (resp. $\eta^2 < \xi^2$). The degenerate states of associated with the eigenvalue $f(n_1, n_2)$ of $\mathcal{F}$ are as follows.

- For $\eta^2 > \xi^2$, $f(n_1, n_2) = (\eta^2 - \xi^2)(n_1 + 1)$

$$|\Psi_\pm\rangle = |\widehat{n_1, n_2}\rangle|\pm\rangle = \begin{cases} V^{-1}|n_1, n_2\rangle|+\rangle \\ V^{-1}|n_1 + 1, n_2\rangle|-\rangle. \end{cases} \tag{108}$$

- For $\eta^2 < \xi^2$, $f(n_1, n_2) = (\xi^2 - \eta^2)(n_2 + 1)$

$$|\Psi_\pm\rangle = |\widehat{n_1, n_2}\rangle|\pm\rangle = \begin{cases} V^{-1}|n_1, n_2 + 1\rangle|+\rangle \\ V^{-1}|n_1, n_2\rangle|-\rangle. \end{cases} \tag{109}$$

Since we have

$$H = E_{31}\left(a_1^+ a_1 - a_2^+ a_2 + S_z\right) + \sqrt{\mathcal{F}}(P_+ + P_-) = E_{31}\Delta + \sqrt{\mathcal{F}}(P_+ + P_-), \tag{110}$$

where Δ commutes with the $su(2)(P)$ generators as well as with the V transformation we are left with a standard problem as in equation (26) with $\delta_z = 0$, $\delta = 1$ and with undressed states as given in equations (108), (109). Straightforward application of the results in section 2 gives

$$\begin{aligned} E_{n_1 n_2 \pm} &= E_{31}\left(n_1 - n_2 + \tfrac{1}{2}\right) \pm (\eta^2 - \xi^2)^{1/2}(n_1 + 1)^{1/2} & \eta^2 > \xi^2, \\ E_{n_1 n_2 \pm} &= E_{31}\left(n_1 - n_2 - \tfrac{1}{2}\right) \pm (\xi^2 - \eta^2)^{1/2}(n_2 + 1)^{1/2} & \eta^2 < \xi^2, \end{aligned} \tag{111}$$

and the eigenstates of H

$$|\widetilde{\Psi}_\pm\rangle = U^{-1}|\widehat{n_1, n_2}\rangle|\pm\rangle, \tag{112}$$

with (equation (21))

$$U^{-1} = \exp\left[-\frac{\pi}{4}(P_+ - P_-)\right] = (I - P_+)\exp[\ln(2)P_z](I + P_-).$$

We note that disentangling the V operator in equation (107) allows us to determine the equivalent field mode operators Va_iV^{-1}, $Va_i^+V^{-1}$ and thus VHV^{-1}. This shows clearly that this transformation performs a complete decoupling of the Stokes and anti-Stokes fields. We find

$$\begin{aligned} VHV^{-1} &= E_{31}(N_1 - N_2 + S_z) + (\eta^2 - \xi^2)^{1/2}\left(a_1 S_+ + a_1^+ S_-\right) & \eta^2 > \xi^2, \\ &= E_{31}(N_1 - N_2 + S_z) + (\xi^2 - \eta^2)^{1/2}\left(a_2^+ S_+ + a_2 S_-\right) & \eta^2 < \xi^2, \end{aligned} \tag{113}$$

which is a standard one-mode JCM on the anti-Stokes field ($i = 1$) when $\eta^2 > \xi^2$ and the equivalent of the counter-rotating terms of a one-mode JCM on the Stokes field ($i = 2$) when $\eta^2 < \xi^2$.

The case $\eta = \xi$ considered in [71] gives

$$\mathcal{F} = \xi^2[2K_z + K_+ + K_-] = 2\xi^2[K_z + K_x] \tag{114}$$

and the V transformation cannot be determined. It has been shown [72, 73] that this corresponds to unnormalizable eigenstates in the sense $\langle\Psi|\Psi\rangle < \infty$ but that can be normalized by the delta function [74]. It is interesting to note that the linear vibronic interaction term in $E \otimes \varepsilon$ Jahn–Teller systems [63, 66] is of the same form as in (104) when $\eta = \xi$ and thus the $\mathcal{F}$ operator is also given by (114). The impossibility to find exact analytical solutions comes for the zeroth-order term which for this system is that of a two-dimensional isotropic harmonic oscillator $H_0 = \hbar\omega\big(a_1^+a_1 + a_2^+a_2 + 1\big) = 2\hbar\omega K_z$.

It may be verified that the technique used in this section gives an alternative to solve the modified two-mode JCM of section 3.3; in this case $\mathcal{F}$ is a function in the generators of an $su(2)$ algebra. However, it is less convenient for a generalization to a p-mode case.

5. Conclusion

In this paper we showed that solvable quantum models involving pseudo-spin interactions may be built from general assumptions concerning the preponderant interaction term. The algebraic structure of this term leads to a hermitian operator $\mathcal{F}$, the form of which determines next the additional terms which may appear in the Hamiltonian expansion. We mainly dealt with three algebras which occur frequently in quantum physics but others could be considered. Other solvable Hamiltonian models, containing up to seven significant parameters, in the areas of rovibrational and rovibronic spectroscopy will be presented elsewhere. In these cases our approach allows, in addition, a straightforward determination of symmetry adapted eigenstates.

Appendix

We gather below some relations useful when dealing with transformations of the $su(2)$ algebra.

A.1. Equivalent elementary operators

We consider transformations of the form

$$U = \exp(\mathrm{i}\theta Z) = \exp\left\{\mathrm{i}\theta\big[f_1a_1^+a_1 + f_2a_2^+a_2 + f_+a_1^+a_2 + f_-a_2^+a_1\big]\right\}, \tag{A.1}$$

for a Schwinger realization (48), (49) of a u_2 algebra. Since $N = N_1 + N_2$ commute with the $su(2)$ generators we may also write (A.1)

$$U = \exp[\mathrm{i}\theta(f_1 + f_2)N/2]\exp\left\{\mathrm{i}\theta\big[(f_1 - f_2)\big(a_1^+a_1 - a_2^+a_2\big)/2 + f_+a_1^+a_2 + f_-a_2^+a_1\big]\right\}. \tag{A.2}$$

The similarity transformation of an element X writes

$$\widetilde{X}(\theta) = UXU^{-1} = \exp(\mathrm{i}\theta Z)X\exp(-\mathrm{i}\theta Z), \tag{A.3}$$

and differentiation with respect to θ leads to

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\widetilde{X}(\theta) = -\mathrm{i}\widetilde{[X, Z]} = -\mathrm{i}[\widetilde{X}, Z]. \tag{A.4}$$

In particular, when X is an elementary boson operator a_i or a_i^+ $(i = 1, 2)$ integration of equation (A.4) gives, setting $\theta = 1$ and $t = [(f_1 - f_2)^2 + 4f_+f_-]^{1/2}$,

$$\widetilde{a}_1 = \exp[-\mathrm{i}(f_1 + f_2)/2]\left\{\cos\left(\frac{t}{2}\right)a_1 - \frac{\mathrm{i}(f_1 - f_2)}{t}\sin\left(\frac{t}{2}\right)a_1 - \frac{2\mathrm{i}f_+}{t}\sin\left(\frac{t}{2}\right)a_2\right\},$$
$$\widetilde{a_1^+} = \exp[\mathrm{i}(f_1 + f_2)/2]\left\{\cos\left(\frac{t}{2}\right)a_1^+ + \frac{\mathrm{i}(f_1 - f_2)}{t}\sin\left(\frac{t}{2}\right)a_1^+ + \frac{2\mathrm{i}f_-}{t}\sin\left(\frac{t}{2}\right)a_2^+\right\}. \tag{A.5}$$

$\widetilde{a}_2$ and $\widetilde{a_2^+}$ are obtained from (A.5) with the interchange $1 \leftrightarrow 2$ and $f_+ \leftrightarrow f_-$. From these transformation laws those for the $su(2)$ generators (48) can be deduced. A similar procedure can be used for the other algebras considered in this paper in their various bosonic realizations.

Table A.1. P_R operators (equations (79), (80)).

(μ, ν, κ)	$\epsilon_{\mu,\nu,\kappa}$	λ	$\vec{\Lambda}$	P_R
x, y, z	1	1	(0, 0, 0)	I
y, x, z	-1	0	$(1, 1, 0)/\sqrt{2}$	$\exp\left[3\mathrm{i}\frac{\pi}{4} J_z\right] \exp[\mathrm{i}\pi J_y] \exp\left[\mathrm{i}\frac{\pi}{4} J_z\right]$
y, z, x	1	1/2	$(-1, -1, -1)/2$	$\exp\left[\mathrm{i}\frac{\pi}{2} J_y\right] \exp\left[\mathrm{i}\frac{\pi}{2} J_z\right]$
z, y, x	-1	$1/\sqrt{2}$	$(0, 1, 0)/\sqrt{2}$	$\exp\left[\mathrm{i}\frac{\pi}{2} J_y\right]$
z, x, y	1	1/2	$(1, 1, 1)/2$	$\exp\left[\mathrm{i}\frac{\pi}{2} J_z\right] \exp\left[\mathrm{i}\frac{\pi}{2} J_y\right]$
x, z, y	-1	$1/\sqrt{2}$	$(-1, 0, 0)/\sqrt{2}$	$\exp\left[- i\frac{\pi}{2} J_z\right] \exp\left[\mathrm{i}\frac{\pi}{2} J_y\right] \exp\left[\mathrm{i}\frac{\pi}{2} J_z\right]$

A.2. Euler–Rodrigues parameters

As discussed in [75–77], the Euler–Rodrigues parameters $(\lambda, \vec{\Lambda})$ present advantages over the more traditional Euler angles for the parametrization of an element of the group $SO(3)$. For a rotation $R(\varphi, \vec{n})$ of angle φ $(0 \leqslant \varphi \leqslant \pi)$ and axis $\vec{n}$, they are defined by

$$\lambda = \cos\frac{\varphi}{2}, \qquad \vec{\Lambda} = \sin\frac{\varphi}{2}\vec{n}, \tag{A.6}$$

and the inverse rotation is $R^{-1}(\varphi, \vec{n}) = R(\varphi, -\vec{n})$. In the *passive* point of view the associated operator is

$$\begin{aligned} P_R &= \exp(\mathrm{i}\varphi\vec{n}\cdot\vec{J}) = \exp[\mathrm{i}\varphi(n_x J_x + n_y J_y + n_z J_z)] \\ &= \exp[\mathrm{i}\varphi(n_- J_+ + n_+ J_- + n_z J_z)] \end{aligned} \tag{A.7}$$

with $n_\pm = (n_x \pm \mathrm{i}n_y)/2$. Using the $su(2)$ disentangling formula [19] the P_R operator is rewritten as

$$P_R = \exp(g_+(\varphi)J_+)\exp(g_z(\varphi)J_z)\exp(g_-(\varphi)J_-), \tag{A.8}$$

which gives

$$g_+(\varphi) = \mathrm{i}\frac{\Lambda_x - \mathrm{i}\Lambda_y}{\lambda - \mathrm{i}\Lambda_z}, \qquad g_-(\varphi) = \mathrm{i}\frac{\Lambda_x + \mathrm{i}\Lambda_y}{\lambda - \mathrm{i}\Lambda_z}, \qquad g_z(\varphi) = \ln[\lambda - \mathrm{i}\Lambda_z]^{-2}. \tag{A.9}$$

The transformation laws of the standard $su(2)$ irreducible bases $\{|jm\rangle\}$ under the action of P_R operators is

$$P_R|jm\rangle = \widehat{|jm\rangle} = \sum_{m'} \mathcal{D}^{(j)*}_{mm'}(\rho, \tau)|jm'\rangle, \tag{A.10}$$

where the rotation matrices are those in [75] expressed in terms of the complex quaternion parameters $\rho = \lambda - \mathrm{i}\Lambda_z$, $\tau = \Lambda_x + \mathrm{i}\Lambda_y$.

A.3. Rotation operators for various reference configurations

We give in table A.1 the unitary operators P_R associated with a change of quantization axis from the standard (x, y, z) configuration to an arbitrary (μ, ν, κ) one (equations (79), (80)). With equations (A.10) they allow the determination of the eigenstates (87), (93), (94) in terms of the standard initial basis $|jm\rangle|\pm\rangle$. They are also given in a form with which the transformed of the $su(2)$ generators are easily obtained.

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